"JUST THE MATHS"

UNIT NUMBER

16.1

LAPLACE TRANSFORMS 1
(Definitions and rules)

by

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UNIT 16.1 - LAPLACE TRANSFORMS 1 - DEFINITIONS AND RULES

16.1.1 INTRODUCTION

The theory of **"Laplace Transforms"** to be discussed in the following notes will be for the purpose of solving certain kinds of **"differential equation"**; that is, an equation which involves a derivative or derivatives.

The particular differential equation problems to be encountered will be limited to the two types listed below:

(a) Given the **"first order linear differential equation with constant coefficients"**,

$$a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = f(t),$$

together with the value of x when $t = 0$ (that is, $x(0)$), determine a formula for x in terms of t, which does not include any derivatives.

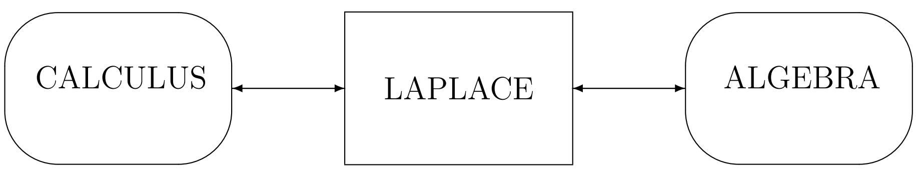
(b) Given the **"second order linear differential equation with constant coefficients"**,

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t),$$

together with the values of x and $\frac{\mathrm{d}x}{\mathrm{d}t}$ when $t = 0$ (that is, $x(0)$ and $x'(0)$), determine a formula for x in terms of t which does not include any derivatives.

Roughly speaking, the method of Laplace Transforms is used to convert a calculus problem (the differential equation) into an algebra problem (frequently an exercise on partial fractions and/or completing the square).

The solution of the algebra problem is then fed backwards through what is called an **"Inverse Laplace Transform"** and the solution of the differential equation is obtained.



The background to the development of Laplace Transforms would be best explained using certain other techniques of solving differential equations which may not have been part of earlier work. This background will therefore be omitted here.

DEFINITION
The Laplace Transform of a given function $f(t)$, defined for $t > 0$, is defined by the definite integral

$$\int_0^\infty e^{-st} f(t)\ \mathrm{d}t,$$

where s is an **arbitrary positive number**.

Notes
(i) The Laplace Transform is usually denoted by $L[f(t)]$ or $F(s)$, since the result of the definite integral in the definition will be an expression involving s.

(ii) Although s is an arbitrary positive number, it is occasionally necessary to assume that it is large enough to avoid difficulties in the calculations; (see the note to the second standard result below).

16.1.2 LAPLACE TRANSFORMS OF SIMPLE FUNCTIONS

The following is a list of standard results on which other Laplace Transforms will be based:

1. $f(t) \equiv t^n$.

$$F(s) = \int_0^\infty e^{-st} t^n\ \mathrm{d}t = I_n \text{ say}.$$

Hence,

$$I_n = \left[\frac{t^n e^{-st}}{-s}\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1}\ \mathrm{d}t = \frac{n}{s}.I_{n-1},$$

using the fact that e^{-st} tends to zero much faster than any other function of t can tend to infinity. That is, a decaying exponential will always have the dominating effect.

We conclude that

$$I_n = \frac{n}{s}.\frac{(n-1)}{s}.\frac{(n-2)}{s}....\frac{2}{s}.\frac{1}{s}.I_0 = \frac{n!}{s^n}.I_0.$$

But,

$$I_0 = \int_0^\infty e^{-st}\ \mathrm{d}t = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}.$$

Thus,

$$L[t^n] = \frac{n!}{s^{n+1}}.$$

Note:
This result also shows that

$$L[1] = \frac{1}{s},$$

since $1 = t^0$.

2. $f(t) \equiv e^{-at}$.

$$F(s) = \int_0^{\infty} e^{-st} e^{-at} \text{ d}t = \int_0^{\infty} e^{-(s+a)t} \text{ d}t$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty} = \frac{1}{s+a}.$$

Hence,

$$L[e^{-at}] = \frac{1}{s+a}.$$

Note:
A slightly different form of this result, less commonly used in applications to science and engineering, is

$$L[e^{bt}] = \frac{1}{s-b};$$

but, to obtain this result by integration, we would need to assume that $s > b$ to ensure that $e^{-(s-b)t}$ is genuinely a **decaying** exponential.

3. $f(t) \equiv \cos at$.

$$F(s) = \int_0^{\infty} e^{-st} \cos at \text{ d}t = \left[\frac{e^{-st} \sin at}{a} \right]_0^{\infty} + \frac{s}{a} \int_0^{\infty} e^{-st} \sin at \text{ d}t$$

using Integration by Parts, once.

Using Integration by Parts a second time,

$$F(s) = 0 + \frac{s}{a} \left\{ \left[-\frac{e^{-st} \cos at}{a} \right]_0^{\infty} - \frac{s}{a} \int_0^{\infty} e^{-st} \cos at \text{ d}t \right\},$$

which gives

$$F(s) = \frac{s}{a^2} - \frac{s^2}{a^2}.F(s).$$

That is,

$$F(s) = \frac{s}{s^2 + a^2}.$$

In other words,

$$L[\cos at] = \frac{s}{s^2 + a^2}.$$

4. $f(t) \equiv \sin at$.

The method is similar to that for $\cos at$, and we obtain

$$L[\sin at] = \frac{a}{s^2 + a^2}.$$

16.1.3 ELEMENTARY LAPLACE TRANSFORM RULES

The following list of results is of use in finding the Laplace Transform of a function which is made up of **basic** functions, such as those encountered in the previous section.

1. LINEARITY

If A and B are constants, then

$$L[Af(t) + Bg(t)] = AL[f(t)] + BL[g(t)].$$

Proof:

This follows easily from the linearity of an integral.

EXAMPLE

Determine the Laplace Transform of the function,

$$2t^5 + 7\cos 4t - 1.$$

Solution

$$L[2t^5 + 7\cos 4t - 1] = 2.\frac{5!}{s^6} + 7.\frac{s}{s^2 + 4^2} - \frac{1}{s} = \frac{240}{s^6} + \frac{7s}{s^2 + 16} - \frac{1}{s}.$$

2. THE TRANSFORM OF A DERIVATIVE

The two results which follow are of special use when solving first and second order differential equations. We shall begin by discussing them in relation to an arbitrary function, $f(t)$; then we shall restate them in the form which will be needed for solving differential equations.

(a)

$$L[f'(t)] = sL[f(t)] - f(0).$$

Proof:

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\ \mathrm{d}t = \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\ \mathrm{d}t$$

using integration by parts.

Thus,

$$L[f'(t)] = -f(0) + sL[f(t)],$$

as required.

(b)

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0).$$

Proof:

Treating $f''(t)$ as the first derivative of $f'(t)$, we have

$$L[f''(t)] = sL[f'(t)] - f'(0),$$

which gives the required result on substituting from (a) the expression for $L[f'(t)]$.

Alternative Forms (Using $L[x(t)] = X(s)$)**:**

(i)

$$L\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right] = sX(s) - x(0).$$

(ii)

$$L\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right] = s^2X(s) - sx(0) - x'(0) \text{ or } s[sX(s) - x(0)] - x'(0).$$

3. THE (First) SHIFTING THEOREM

$$L[e^{-at}f(t)] = F(s+a).$$

Proof:

$$L[e^{-at}f(t)] = \int_0^\infty e^{-st}e^{-at}f(t)\ \mathrm{d}t = \int_0^\infty e^{-(s+a)t}f(t)\ \mathrm{d}t,$$

which can be regarded as the effect of replacing s by $s+a$ in $L[f(t)]$. In other words, $F(s+a)$.

Notes:

(i) Sometimes, this result is stated in the form

$$L[e^{bt}f(t)] = F(s-b)$$

but, in science and engineering, the exponential is more likely to be a **decaying** exponential.

(ii) There is, in fact, a Second Shifting Theorem, encountered in more advanced courses; but we do not include it in this Unit (see Unit 16.5).

EXAMPLE

Determine the Laplace Transform of the function, $e^{-2t}\sin 3t$.

Solution

First of all, we note that

$$L[\sin 3t] = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}.$$

Replacing s by $(s+2)$ in this result, the First Shifting Theorem gives

$$L[e^{-2t}\sin 3t] = \frac{3}{(s+2)^2+9}.$$

4. MULTIPLICATION BY t

$$L[tf(t)] = -\frac{\mathrm{d}}{\mathrm{d}s}[F(s)].$$

Proof:

It may be shown that

$$\frac{\mathrm{d}}{\mathrm{d}s}[F(s)] = \int_0^\infty \frac{\partial}{\partial s}[e^{-st}f(t)]\mathrm{d}t = \int_0^\infty -te^{-st}f(t)\ \mathrm{d}t = -\ L[tf(t)].$$

EXAMPLE

Determine the Laplace Transform of the function,

$$t\cos 7t.$$

Solution

$$L[t\cos 7t] = -\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{s}{s^2+7^2}\right] = -\ \frac{(s^2+7^2).1 - s.2s}{(s^2+7^2)^2} = \frac{s^2-49}{(s^2+49)^2}.$$

THE USE OF A TABLE OF LAPLACE TRANSFORMS AND RULES

For the purposes of these Units, the following **brief** table may be used to determine the Laplace Transforms of functions of t without having to use integration:

$f(t)$	$L[f(t)] = F(s)$
K (a constant)	$\frac{K}{s}$
e^{-at}	$\frac{1}{s+a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\text{sinh}at$	$\frac{a}{s^2-a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\text{cosh}at$	$\frac{s}{s^2-a^2}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$
$t\cos at$	$\frac{(s^2-a^2)}{(s^2+a^2)^2}$
$\sin at - at\cos at$	$\frac{2a^3}{(s^2+a^2)^2}$

16.1.4 FURTHER LAPLACE TRANSFORM RULES

1.

$$L\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right] = sX(s) - x(0).$$

2.

$$L\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right] = s^2X(s) - sx(0) - x'(0) \quad \text{or} \quad s[sX(s) - x(0)] - x'(0).$$

3. **The Initial Value Theorem**

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s),$$

provided that the indicated limits exist.

4. **The Final Value Theorem**

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s),$$

provided that the indicated limits exist.

5. **The Convolution Theorem**

$$L\left[\int_0^t f(T)g(t-T)\ \mathrm{d}T\right] = F(s)G(s).$$

16.1.5 EXERCISES

1. Use a table the table of Laplace Transforms to find $L[f(t)]$ in the following cases:

   (a)
$$3t^2 + 4t - 1;$$

   (b)
$$t^3 + 3t^2 + 3t + 1 \quad (\equiv (t+1)^3);$$

   (c)
$$2e^{5t} - 3e^t + e^{-7t};$$

   (d)
$$2\sin 3t - 3\cos 2t;$$

   (e)
$$t\sin 6t;$$

   (f)
$$t(e^t + e^{-2t});$$

   (g)
$$\frac{1}{2}(1 - \cos 2t) \quad (\equiv \sin^2 t).$$

2. Using the First Shifting Theorem, obtain the Laplace Transforms of the following functions of t:

   (a)
$$e^{-3t}\cos 5t;$$

   (b)
$$t^2 e^{2t};$$

(c)

$$e^{-2t}\left(2t^3+3t-2\right);$$

(d)

$$\cosh 2t.\sin t;$$

(e)

$$e^{-at}f'(t),$$

where $L[f(t)]=F(s)$.

3. (a) If

$$x=t^3e^{-t},$$

determine the Laplace Transform of $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ without differentiating x more than once with respect to t.

(b) If

$$\frac{\mathrm{d}x}{\mathrm{d}t}+x=e^t,$$

where $x(0)=0$, show that

$$X(s)=\frac{1}{s^2-1}.$$

4. Verify the Initial and Final Value Theorems for the function

$$f(t)=te^{-3t}.$$

16.1.6 ANSWERS TO EXERCISES

1. (a)

$$\frac{6}{s^3}+\frac{4}{s^2}-\frac{1}{s};$$

(b)

$$\frac{6}{s^4}+\frac{6}{s^3}+\frac{3}{s^2}+\frac{1}{s};$$

(c)

$$\frac{2}{s-5}-\frac{3}{s-1}+\frac{1}{s+7};$$

(d)

$$\frac{6}{s^2+9}-\frac{3s}{s^2+4};$$

(e)

$$\frac{12s}{(s^2+36)^2};$$

(f)

$$\frac{1}{(s-1)^2}+\frac{1}{(s+2)^2};$$

(g)

$$\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4}\right].$$

2. (a)

$$\frac{s+3}{(s+3)^2+25};$$

(b)

$$\frac{2}{(s-2)^3};$$

(c)

$$\frac{12}{(s+2)^4}+\frac{3}{(s+2)^2}-\frac{2}{s+2};$$

(d)

$$\frac{1}{2}\left[\frac{1}{(s-2)^2+1}+\frac{1}{(s+2)^2+1}\right];$$

(e)

$$(s+a)F(s+a)-f(0).$$

3. (a)

$$\frac{6s^2}{(s+1)^4};$$

(b) On the left hand side, use the formula for $L\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right]$.

4.

$$\lim_{t\to 0} f(t)=0 \;\text{ and }\; \lim_{t\to\infty} f(t)=0.$$