"JUST THE MATHS"

UNIT NUMBER

15.8

ORDINARY DIFFERENTIAL EQUATIONS 8 (Simultaneous equations (A))

by

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UNIT 15.8 - ORDINARY DIFFERENTIAL EQUATIONS 8

SIMULTANEOUS EQUATIONS (A)

15.8.1 THE SUBSTITUTION METHOD

The methods discussed in previous Units for the solution of second order ordinary linear differential equations with constant coefficients may now be used for cases of two first order differential equations which must be satisfied simultaneously. The technique will be illustrated by the following examples:

EXAMPLES

1. Determine the general solutions for y and z in the case when

$$\begin{aligned} 5\frac{\mathrm{d}y}{\mathrm{d}x} - 2\frac{\mathrm{d}z}{\mathrm{d}x} + 4y - z &= e^{-x}, - - - - - - - - - - - (1) \\ \frac{\mathrm{d}y}{\mathrm{d}x} + 8y - 3z &= 5e^{-x}. - - - - - - - - - - (2) \end{aligned}$$

Solution

First, we eliminate one of the dependent variables from the two equations; in this case, we eliminate z.

From equation (2),

$$z = \frac{1}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}x} + 8y - 5e^{-x}\right)$$

and, on substituting this into equation (1), we obtain

$$5\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{3}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 8\frac{\mathrm{d}y}{\mathrm{d}x} + 5e^{-x}\right) + 4y - \frac{1}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}x} + 8y - 5e^{-x}\right) = e^{-x}.$$

That is, $$-\frac{2}{3}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{2}{3}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{4}{3}y = \frac{8}{3}e^{-x}$$

or

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = -4e^{-x}.$$

The auxiliary equation is

$$m^2 + m - 2 = 0 \quad \text{or} \quad (m-1)(m+2) = 0,$$

giving a complementary function of $Ae^x + Be^{-2x}$, where A and B are arbitrary constants. A particular integral will be of the form ke^{-x}, where $k - k - 2k = -4$ and hence $k = 2$. Thus,

$$y = 2e^{-x} + Ae^x + Be^{-2x}.$$

Finally, from the formula for z in terms of y,

$$z = \frac{1}{3}\left(-2e^{-x} + Ae^x - 2Be^{-2x} + 16e^{-x} + 8Ae^x + 8Be^{-2x} - 5e^{-x}\right).$$

That is,

$$z = 3e^{-x} + 3Ae^{-x} + 2Be^{-2x}.$$

Note:
The above example would have been a little more difficult if the second differential equation had contained a term in $\frac{\mathrm{d}z}{\mathrm{d}x}$. But, if this were the case, we could eliminate $\frac{\mathrm{d}z}{\mathrm{d}x}$ between the two equations in order to obtain a statement with the same form as Equation (2).

2. Solve, simultaneously, the differential equations

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}x} + 2y &= e^x, - - - - - - - - - (1) \\ \frac{\mathrm{d}y}{\mathrm{d}x} - 2z &= 1 + x, - - - - - - - -(2) \end{aligned}$$

given that $y = 1$ and $z = 2$ when $x = 0$.

Solution:

From equation (2), we have

$$z = \frac{1}{2}\left[\frac{\mathrm{d}y}{\mathrm{d}x} - 1 - x\right].$$

Substituting into the first differential equation gives

$$\frac{1}{2}\left[\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 1\right] + 2y = e^x$$

or

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = 2e^x + 1.$$

The auxiliary equation is therefore $m^2 + 4 = 0$, having solutions $m = \pm j2$, which means that the complementary function is

$$A\cos 2x + B\sin 2x,$$

where A and B are arbitrary constants.
The particular integral will be of the form $y = pe^x + q$,
where

$$pe^x + 4pe^x + 4q = 2e^x + 1.$$

We require, then, that $5p = 2$ and $4q = 1$; and so the general solution for y is

$$y = A\cos 2x + B\sin 2x + \frac{2}{5}e^x + \frac{1}{4}.$$

Using the earlier formula for z, we obtain

$$z = \frac{1}{2}\left[-2A\sin 2x + 2B\cos 2x + \frac{2}{5}e^x - 1 - x\right] = B\cos 2x - A\sin 2x + \frac{1}{5}e^x - \frac{1}{2} - \frac{x}{2}.$$

Applying the boundary conditions,

$$1 = A + \frac{2}{5} + \frac{1}{4} \ \text{ giving } \ A = \frac{7}{20}$$

and

$$2 = B + \frac{1}{5} - \frac{1}{2} \ \text{ giving } \ B = \frac{23}{10}.$$

The required solutions are therefore

$$y = \frac{7}{20}\cos 2x + \frac{23}{10}\sin 2x + \frac{2}{5}e^x + \frac{1}{4}$$

and

$$z = \frac{23}{10}\cos 2x - \frac{7}{20}\sin 2x + \frac{1}{5}e^x - \frac{1}{2} - \frac{x}{2}.$$

15.8.2 EXERCISES

Solve the following pairs of simultaneous differential equations, subject to the given boundary conditions:

1. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + 2z &= e^{-x},\\ \frac{\mathrm{d}z}{\mathrm{d}x} + 3z &= y,\end{aligned}$$

   given that $y = 1$ and $z = 0$ when $x = 0$.

2. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} - z &= \sin x,\\ \frac{\mathrm{d}z}{\mathrm{d}x} + y &= \cos x,\end{aligned}$$

   given that $y = 3$ and $z = 4$ when $x = 0$.

3. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + 2y - 3z &= 1,\\ \frac{\mathrm{d}z}{\mathrm{d}x} - y &= e^{-2x},\end{aligned}$$

   given that $y = 0$ and $z = 0$ when $x = 0$.

4. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2z,\\ \frac{\mathrm{d}z}{\mathrm{d}x} &= 8y,\end{aligned}$$

   given that $y = 1$ and $z = 0$ when $x = 0$.

5.

$$\begin{aligned}\frac{dy}{dx}+4\frac{dz}{dx}+6z &= 0,\\ 5\frac{dy}{dx}+2\frac{dz}{dx}+6y &= 0,\end{aligned}$$

given that $y = 3$ and $z = 0$ when $x = 0$.
Hint: First eliminate the $\frac{dz}{dx}$ terms to obtain a formula for z in terms of y and $\frac{dy}{dx}$.

6.

$$\begin{aligned}10\frac{dy}{dx}-3\frac{dz}{dx}+6y+5z &= 0,\\ 2\frac{dy}{dx}-\frac{dz}{dx}+2y+z &= 2e^{-x},\end{aligned}$$

given that $y = 2$ and $z = -1$ when $x = 0$.
Hint: First, eliminate the $\frac{dy}{dx}$ and z terms in one step, to obtain a formula for y in terms of $\frac{dz}{dx}$ and x.

15.8.3 ANSWERS TO EXERCISES

1.

$$y = (2x+1)e^{-x} \quad \text{and} \quad z = xe^{-x}.$$

2.

$$y = (x+4)\sin x + 3\cos x \quad \text{and} \quad z = (x+4)\cos x - 3\sin x.$$

3.

$$y = \frac{1}{2}e^{x} + \frac{1}{2}e^{-3x} - e^{-2x} \quad \text{and} \quad z = \frac{1}{2}e^{x} - \frac{1}{6}e^{-3x} - \frac{1}{3}.$$

4.

$$y = \frac{1}{2}e^{4x} - \frac{1}{2}e^{-4x} \equiv \sinh 4x \quad \text{and} \quad z = e^{4x} + e^{-4x} \equiv 2\cosh 4x.$$

5.

$$y = 2e^{-x} + e^{-2x} \quad \text{and} \quad z = e^{-x} - e^{-2x}.$$

6.

$$y = \sin x + 2e^{-x} \quad \text{and} \quad z = e^{-x} - 2\cos x.$$