"JUST THE MATHS"

UNIT NUMBER

15.7

ORDINARY DIFFERENTIAL EQUATIONS 7 (Second order equations (D))

by

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UNIT 15.7 - ORDINARY DIFFERENTIAL EQUATIONS 7

SECOND ORDER EQUATIONS (D)

15.7.1 PROBLEMATIC CASES OF PARTICULAR INTEGRALS

Difficulties can arise if all or part of any trial solution would already be included in the complementary function. We illustrate with some examples:

EXAMPLES

1. Determine the complementary function and a particular integral for the differential equation
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{2x}.$$

Solution

The auxiliary equation is $m^2 - 3m + 2 = 0$, with solutions $m = 1$ and $m = 2$ and hence the complementary function is $Ae^x + Be^{2x}$, where A and B are arbitrary constants.

A trial solution of $y = \alpha e^{2x}$ gives
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\alpha e^{2x} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4\alpha e^{2x}$$
and, on substituting these into the differential equation, it is necessary that
$$4\alpha e^{2x} - 6\alpha e^{2x} + 2\alpha e^{2x} \equiv e^{2x}.$$
That is, $0 \equiv e^{2x}$ which is impossible.

However, if $y = \alpha e^{2x}$ has proved to be unsatisfactory, let us investigate, as an alternative, $y = F(x)e^{2x}$ (where $F(x)$ is a function of x instead of a constant).

We have
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2F(x)e^{2x} + F'(x)e^{2x}$$
and, hence,
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4F(x)e^{2x} + 2F'(x)e^{2x} + F''(x)e^{2x} + 2F'(x)e^{2x}.$$

On substituting these into the differential equation, it is necessary that

$$(4F(x) + 2F'(x) + F''(x) + 2F'(x) - 6F(x) - 3F'(x) + 2F(x))\, e^{2x} \equiv e^{2x}.$$

That is,

$$F''(x) + F'(x) = 1,$$

which is satisfied by the function $F(x) \equiv x$ and thus a suitable particular integral is

$$y = xe^{2x}.$$

Note:
It may be shown, in other cases too that, if the standard trial solution is already contained in the complementary function, then it is is necessary to multiply it by x in order to obtain a suitable particular integral.

2. Determine the complementary function and a particular integral for the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = \sin x.$$

Solution

The auxilary equation is $m^2 + 1 = 0$, with solutions $m = \pm j$ and, hence, the complementary function is $A \sin x + B \cos x$, where A and B are arbitrary constants.

A trial solution of $y = \alpha \sin x + \beta \cos x$ gives

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\alpha \sin x - \beta \cos x;$$

and, on substituting into the differential equation, it is necessary that $0 \equiv \sin x$, which is impossible.

Here, we may try $y = x(\alpha \sin x + \beta \cos x)$, giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha \sin x + \beta \cos x + x(\alpha \cos x - \beta \sin x) = (\alpha - \beta x) \sin x + (\beta + \alpha x) \cos x$$

and, therefore,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (\alpha - \beta x) \cos x - \beta \sin x - (\beta + \alpha x) \sin x + \alpha \cos x = (2\alpha - \beta x) \cos x - (2\beta + \alpha x) \sin x.$$

Substituting into the differential equation, we thus require that

$$(2\alpha - \beta x)\cos x - (2\beta + \alpha x)\sin x + x(\alpha \sin x + \beta \cos x) \equiv \sin x,$$

which simplifies to

$$2\alpha \cos x - 2\beta \sin x \equiv \sin x.$$

Thus $2\alpha = 0$ and $-2\beta = 1$.

An appropriate particular integral is now

$$y = -\frac{1}{2}x\cos x.$$

3. Determine the complementary function and a particular integral for the differential equation

$$9\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + y == 50e^{-\frac{1}{3}x}.$$

Solution

The auxiliary equation is $9m^2 + 6m + 1 = 0$, or $(3m+1)^2 = 0$, which has coincident solutions $m = -\frac{1}{3}$ and so the complementary function is

$$(Ax + B)e^{-\frac{1}{3}x}.$$

In this example, both $e^{-\frac{1}{3}x}$ **and** $xe^{-\frac{1}{3}x}$ are contained in the complementary function.

Thus, in the trial solution, it is necessary to multiply by a **further** x, giving

$$y = \alpha x^2 e^{-\frac{1}{3}x}.$$

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\alpha x e^{-\frac{1}{3}x} - \frac{1}{3}x^2 e^{\frac{1}{3}x}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\alpha e^{-\frac{1}{3}x} - \frac{2}{3}\alpha x e^{-\frac{1}{3}x} - \frac{2}{3}\alpha x e^{-\frac{1}{3}x} + \frac{1}{9}\alpha x^2 e^{-\frac{1}{3}x}.$$

Substituting these into the differential equation, it is necessary that

$$\left(18\alpha - 12\alpha x + \alpha x^2 + 12\alpha x - 2\alpha x^2 + \alpha x^2\right) e^{-\frac{1}{3}x} = 50e^{-\frac{1}{3}x}$$

and, hence, $18\alpha = 50$ or $\alpha = \frac{25}{9}$.
An appropriate particular integral is

$$y = \frac{25}{9}x^2 e^{-\frac{1}{3}x}.$$

4. Determine the complementary function and a particular integral for the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = \sinh 2x.$$

Solution

The auxiliary equation is $m^2 - 5m + 6 = 0$ or $(m-2)(m-3) = 0$ which has solutions $m = 2$ and $m = 3$ and, hence, the complementary function is

$$Ae^{2x} + Be^{3x}.$$

However, since $\sinh 2x \equiv \frac{1}{2}(e^{2x} - e^{-2x})$, **part** of it is contained in the complementary function and we must find a particular integral for each part separately.
(a) For $\frac{1}{2}e^{2x}$, we may try

$$y = x\alpha e^{2x},$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha e^{2x} + 2x\alpha e^{2x}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\alpha e^{2x} + 2\alpha e^{2x} + 4x\alpha e^{2x}.$$

Substituting these into the differential equation, it is necessary that

$$(4\alpha + 4x\alpha - 5\alpha - 10x\alpha + 6x\alpha)\, e^{2x} \equiv \frac{1}{2}e^{2x},$$

which gives $\alpha = -\frac{1}{2}$.

(b) For $-\frac{1}{2}e^{-2x}$, we may try

$$y = \beta e^{-2x},$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2\beta e^{-2x}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4\beta e^{-2x}.$$

Substituting these into the differential equation, it is necessary that

$$(4\beta + 10\beta + 6\beta)e^{-2x} \equiv -\frac{1}{2}e^{-2x},$$

which gives $\beta = -\frac{1}{40}$.
The overall particular integral is thus

$$y = -\frac{1}{2}xe^{2x} - \frac{1}{40}e^{-2x}.$$

15.7.2 EXERCISES

Solve completely the following differential equations subject to the given boundary conditions:

1.
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = e^{-x},$$
where $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5}{2}$ when $x = 0$.

2.
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 9y = 2\sin 3x,$$
where $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{8}{3}$ when $x = 0$.

3.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 10\frac{\mathrm{d}y}{\mathrm{d}x} + 25y = 8e^{3x} + 25x^2 - 20x + 27,$$

where $y = 5$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 13$ when $x = 0$.

4.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \cosh x,$$

where $y = \frac{7}{12}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}$ when $x = 0$.

5.

$$4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + y = 24e^{-\frac{1}{2}x}$$

where $y = 6$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 0$.

15.7.3 ANSWERS TO EXERCISES

1.

$$y = \frac{1}{2}xe^{-x} + Ae^{-x} + Be^{-3x}.$$

2.

$$y = -\frac{1}{3}x\cos 3x + 2\cos 3x + \sin 3x.$$

3.

$$y = 2e^{3x} + x^2 + 1 + (2 - 3x)e^{5x}.$$

4.

$$y = \frac{1}{12}\left(e^{-x} - 6xe^{x} - e^{x} + 7e^{2x}\right).$$

5.

$$y = 3x^2e^{-\frac{1}{2}x} + (5x + 6)e^{-\frac{1}{2}x}.$$