"JUST THE MATHS"

UNIT NUMBER

15.6

ORDINARY DIFFERENTIAL EQUATIONS 6 (Second order equations (C))

by

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UNIT 15.6 - ORDINARY DIFFERENTIAL EQUATIONS 6

SECOND ORDER EQUATIONS (C)

15.6.1 RECAP

For the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

it was seen, in Unit 15.5, that

(a) when $f(x) \equiv k$, a given **constant**, a particular integral is $y = \frac{k}{c}$;

(b) when $f(x) \equiv px + q$, a **linear** function in which p and q are given constants, it is possible to obtain a particular integral by assuming that y also has the form of a linear function; that is, we make a **"trial solution"**, $y = \alpha x + \beta$.

15.6.2 FURTHER TYPES OF PARTICULAR INTEGRAL

We now examine particular integrals for other cases of $f(x)$, the method being illustrated by examples. Also, for reasons relating to certain problematic cases discussed in Unit 15.7, we shall determine the complementary function **before** determining the particular integral.

1. $f(x) \equiv px^2 + qx + r$, a **quadratic** function in which p, q and r are given constants; $p \neq 0$.

   $$\text{Trial solution :} \quad y = \alpha x^2 + \beta x + \gamma.$$

   Note:
   This is the trial solution even if q or r (or both) are zero.

   EXAMPLE
   Determine the general solution of the differential equation

   $$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 7\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 4x^2 + 10x - 23.$$

   Solution
   The auxilary equation is

   $$2m^2 - 7m - 4 = 0 \;\text{ or }\; (2m+1)(m-4) = 0,$$

having solutions $m = 4$ and $m = -\frac{1}{2}$.
Thus, the complementary function is

$$Ae^{4x} + Be^{-\frac{1}{2}x},$$

where A and B are arbitrary constants.
To determine a particular integral, we may make a trial solution of the form $y = \alpha x^2 + \beta x + \gamma$, giving $\frac{\mathrm{d}y}{\mathrm{d}x} = 2\alpha x + \beta$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\alpha$.
We thus require that

$$4\alpha - 14\alpha x - 7\beta - 4\alpha x^2 - 4\beta x - 4\gamma \equiv 4x^2 + 10x - 23.$$

That is,

$$-4\alpha x^2 - (14\alpha + 4\beta)x + 4\alpha - 7\beta - 4\gamma \equiv 4x^2 + 10x - 23.$$

Comparing corresponding coefficents on both sides, this means that

$$-4\alpha = 4, \quad -(14\alpha + 4\beta) = 10 \ \text{ and } \ 4\alpha - 7\beta - 4\gamma = -23,$$

which give $\alpha = -1$, $\beta = 1$ and $\gamma = 3$.
Hence, the particular integral is

$$y = 3 + x - x^2.$$

Finally, the general solution is

$$y = 3 + x - x^2 + Ae^{4x} + Be^{-\frac{1}{2}x}.$$

2. $f(x) \equiv p\sin kx + q\cos kx$, a **trigonometric** function in which p, q and k are given constants.

$$\text{Trial solution :} \quad y = \alpha\sin kx + \beta\cos kx.$$

Note:
This is the trial solution even if p or q is zero.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 8\cos 3x - 19\sin 3x.$$

Solution

The auxiliary equation is

$$m^2 - 2m + 2 = 0,$$

which has complex number solutions given by

$$m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm j.$$

Hence, the complementary function is

$$e^x(A\cos x + B\sin x),$$

where A and B are arbitrary constants.

To determine a particular integral, we may make a trial solution of the form

$$y = \alpha\sin 3x + \beta\cos 3x,$$

giving $\frac{\mathrm{d}y}{\mathrm{d}x} = 3\alpha\cos 3x - 3\beta\sin 3x$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -9\alpha\sin 3x - 9\beta\cos 3x$.

We thus require that

$$-9\alpha\sin 3x - 9\beta\cos 3x - 6\alpha\cos 3x + 6\beta\sin 3x + 2\alpha\sin 3x + 2\beta\cos 3x \equiv 8\cos 3x - 19\sin 3x.$$

That is,

$$(-9\alpha + 6\beta + 2\alpha)\sin 3x + (-9\beta - 6\alpha + 2\beta)\cos 3x \equiv 8\cos 3x - 19\sin 3x.$$

Comparing corresponding coefficients on both sides, we have

$$\begin{aligned} -7\alpha + 6\beta &= -19, \\ -6\alpha - 7\beta &= 8. \end{aligned}$$

These equations are satisfied by $\alpha = 1$ and $\beta = -2$, so that the particular integral is

$$y = \sin 3x - 2\cos 3x.$$

Finally, the general solution is

$$y = \sin 3x - 2\cos 3x + e^x(A\cos x + B\sin x).$$

3. $f(x) \equiv pe^{kx}$, an **exponential** function in which p and k are given constants.

$$\text{Trial solution :} \quad y = \alpha e^{kx}.$$

EXAMPLE

Determine the general solution of the differential equation

$$9\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + y = 50e^{3x}.$$

Solution

The auxiliary equation is

$$9m^2 + 6m + 1 = 0 \quad \text{or} \quad (3m+1)^2 = 0,$$

which has coincident solutions at $m = -\frac{1}{3}$.
The complementary function is therefore

$$(Ax + B)e^{-\frac{1}{3}x}.$$

To find a particular integral, we may make a trial solution of the form

$$y = \alpha e^{3x},$$

which gives $\frac{\mathrm{d}y}{\mathrm{d}x} = 3\alpha e^{3x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 9\alpha e^{3x}$.
Hence, on substituting into the differential equation, it is necessary that

$$81\alpha e^{3x} + 18\alpha e^{3x} + \alpha e^{3x} = 50e^{3x}.$$

That is, $100\alpha = 50$, from which we deduce that $\alpha = \frac{1}{2}$ and a particular integral is

$$y = \frac{1}{2}e^{3x}.$$

Finally, the general solution is

$$y = \frac{1}{2}e^{3x} + (Ax + B)e^{-\frac{1}{3}x}.$$

4. $f(x) \equiv p\sinh kx + q\cosh kx$, a **hyperbolic** function in which p, q and k are given constants.

$$\text{Trial solution :} \quad y = \alpha\sinh kx + \beta\cosh kx.$$

Note:
This is the trial solution even if p or q is zero.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 93\cosh 5x - 75\sinh 5x.$$

Solution

The auxiliary equation is

$$m^2 - 5m + 6 = 0 \ \text{ or } \ (m-2)(m-3) = 0,$$

which has solutions $m = 2$ and $m = 3$ so that the complementary function is

$$Ae^{2x} + Be^{3x},$$

where A and B are arbitrary constants.
To determine a particular integral, we may make a trial solution of the form

$$y = \alpha\sinh 5x + \beta\cosh 5x,$$

giving $\frac{\mathrm{d}y}{\mathrm{d}x} = 5\alpha\cosh 5x + 5\beta\sinh 5x$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 25\alpha\sinh 5x + 25\beta\cosh 5x$.
Substituting into the differential equation, the left-hand-side becomes

$$25\alpha\sinh 5x + 25\beta\cosh 5x - 25\alpha\cosh 5x - 25\beta\sinh 5x + 6\alpha\sinh 5x + 6\beta\cosh 5x.$$

This simplifies to

$$(31\alpha - 25\beta)\sinh 5x + (31\beta - 25\alpha)\cosh 5x,$$

so that we require

$$\begin{aligned} 31\alpha - 25\beta &= -75, \\ -25\alpha + 31\beta &= 93, \end{aligned}$$

and these are satisfied by $\alpha = 0$ and $\beta = 3$.

The particular integral is thus

$$y = 3\cosh 5x$$

and, hence, the general solution is

$$y = 3\cosh 5x + Ae^{2x} + Be^{3x}.$$

5. **Combinations of Different Types of Function**

   In cases where $f(x)$ is the sum of two or more of the various types of function discussed previously, then the particular integrals for each type (determined separately) may be added together to give an overall particular integral.

15.6.3 EXERCISES

1. Determine the general solution for each of the following differential equations:

   (a)
   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 4x^2 + 2x - 4;$$

   (b)
   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 8\cos 2x - \sin 2x;$$

   (c)
   $$4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 12\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 27e^{-x};$$

   (d)
   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} - 10y = \cosh 3x - \sinh 3x.$$

2. Solve completely the following differential equations subject to the given boundary conditions:

(a)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - y = 10 - 5x^2 - x + 16e^{-3x},$$

where $y = 13$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -2$ when $x = 0$;

(b)

$$4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 8\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 9x + 6\cos x - 19\sin x,$$

where $y = -2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$.

15.6.4 ANSWERS TO EXERCISES

1. (a)

$$y = x^2 - 2x + 1 + Ae^{-x} + Be^{-4x};$$

(b)

$$y = \sin 2x + e^{-2x}(A\cos x + B\sin x);$$

(c)

$$y = 3e^{-x} + (Ax + B)e^{-\frac{3}{2}x};$$

(d)

$$y = \frac{1}{8}(\cosh 3x - \sinh 3x) + Ae{-}2x + Be^{5x}.$$

2. (a)

$$y = 5x^2 + x + 2e^{-3x} + 3e^{x} - 2e^{-x};$$

(b)

$$y = 3x - 8 + 2\cos x + \sin x + 2e^{-\frac{1}{2}x} + 2e^{-\frac{3}{2}x}.$$