"JUST THE MATHS"

UNIT NUMBER

15.5

ORDINARY DIFFERENTIAL EQUATIONS 5 (Second order equations (B))

by

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UNIT 15.5 - ORDINARY DIFFERENTIAL EQUATIONS 5

SECOND ORDER EQUATIONS (B)

15.5.1 NON-HOMOGENEOUS DIFFERENTIAL EQUATIONS

The following discussion will examine the solution of the second order linear differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

in which a, b and c are constants, but $f(x)$ is not identically equal to zero.

The Particular Integral and Complementary Function

(i) Suppose that $y = u(x)$ is any particular solution of the differential equation; that is, it contains no arbitrary constants. In the present context, we shall refer to such particular solutions as **"particular integrals"** and systematic methods of finding them will be discussed later.

It follows that

$$a\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + b\frac{\mathrm{d}u}{\mathrm{d}x} + cu = f(x).$$

(ii) Suppose also that we make the substitution $y = u(x) + v(x)$ in the original differential equation to give

$$a\frac{\mathrm{d}^2(u+v)}{\mathrm{d}x^2} + b\frac{\mathrm{d}(u+v)}{\mathrm{d}x} + c(u+v) = f(x).$$

That is,

$$a\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + b\frac{\mathrm{d}u}{\mathrm{d}x} + cu + a\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + b\frac{\mathrm{d}v}{\mathrm{d}x} + cv = f(x);$$

and, hence,

$$a\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + b\frac{\mathrm{d}v}{\mathrm{d}x} + cv = 0.$$

This means that the function $v(x)$ is the general solution of the homogeneous differential equation whose auxiliary equation is

$$am^2 + bm + c = 0.$$

In future, $v(x)$ will be called the **"complementary function"** in the general solution of the original (non-homogeneous) differential equation. It complements the particular integral to provide the general solution.

Summary

General solution = particular integral + complementary function.

15.5.2 DETERMINATION OF SIMPLE PARTICULAR INTEGRALS

(a) Particular integrals, when $f(x)$ is a constant, k.

For the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = k,$$

it is easy to see that a particular integral will be $y = \frac{k}{c}$, since its first and second derivatives are both zero, while $cy = k$.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 7\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 20.$$

Solution

(i) By inspection, we may observe that a particular integral is $y = 2$.

(ii) The auxiliary equation is

$$m^2 + 7m + 10 = 0 \quad \text{or} \quad (m + 2)(m + 5) = 0,$$

having solutions $m = -2$ and $m = -5$.

(iii) The complementary function is

$$Ae^{-2x} + Be^{-5x},$$

where A and B are arbitrary constants.

(iv) The general solution is

$$y = 2 + Ae^{-2x} + Be^{-5x}.$$

(b) Particular integrals, when $f(x)$ is of the form $px + q$.

For the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = px + q,$$

it is possible to determine a particular integral by assuming one which has the same form as the right hand side; that is, in this case, another expression consisting of a multiple of x and constant term. The method is, again, illustrated by an example.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 11\frac{\mathrm{d}y}{\mathrm{d}x} + 28y = 84x - 5.$$

Solution

(i) First, we assume a particular integral of the form

$$y = \alpha x + \beta,$$

which implies that $\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$.

Substituting into the differential equation, we require that

$$-11\alpha + 28(\alpha x + \beta) \equiv 84x - 5.$$

Hence, $28\alpha = 84$ and $-11\alpha + 28\beta = -5$, giving $\alpha = 3$ and $\beta = 1$.

Thus, the particular integral is

$$y = 3x + 1.$$

(ii) The auxiliary equation is

$$m^2 - 11m + 28 = 0 \quad \text{or} \quad (m-4)(m-7) = 0,$$

having solutions $m = 4$ and $m = 7$.

(iii) The complementary function is

$$Ae^{4x} + Be^{7x},$$

where A and B are arbitrary constants.

(iv) The general solution is

$$y = 3x + 1 + Ae^{4x} + Be^{7x}.$$

Note:
In examples of the above types, the complementary function must not be prefixed by "$y =$", since the given differential equation, as a whole, is not normally satisfied by the complementary function alone.

15.5.3 EXERCISES

1. Determine the general solutions of the following differential equations:

   (a)
$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 6;$$

   (b)
$$\frac{d^2y}{dx^2} + 16y = 7;$$

   (c)
$$3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y = x + 1;$$

   (d)
$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 18x + 28.$$

2. Solve, completely, the following differential equations, subject to the given boundary condidtions:

   (a)
$$2\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 4y = 100,$$
   where $y = -26$ and $\frac{dy}{dx} = 5$ when $x = 0$;

   (b)
$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 12x + 16,$$
   where $y = 0$ and $\frac{dy}{dx} = 4$ when $x = 0$;

   (c)
$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 10y = 10x + 14,$$
   where $y = 3$ and $\frac{dy}{dx} = 2$ when $x = 0$.

15.5.4 ANSWERS TO EXERCISES

1. (a)

$$y = -3 + Ae^{x} + Be^{2x};$$

(b)

$$y = \frac{7}{16} + A\cos 4x + B\sin 4x;$$

(c)

$$y = 1 - x + Ae^{x} + Be^{-\frac{1}{3}x};$$

(d)

$$y = 2x + 5 + (Ax + B)e^{3x}.$$

2. (a)

$$y = -25 + e^{4x} - 2e^{\frac{1}{2}x};$$

(b)

$$y = 3x + 1 - (x + 1)e^{-2x};$$

(c)

$$y = x + 2 + e^{3x}(\cos x - 2\sin x).$$