"JUST THE MATHS"

UNIT NUMBER

15.4

ORDINARY DIFFERENTIAL EQUATIONS 4 (Second order equations (A))

by

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UNIT 15.4 - ORDINARY DIFFERENTIAL EQUATIONS 4

SECOND ORDER EQUATIONS (A)

15.4.1 INTRODUCTION

In the discussion which follows, we shall consider a particular kind of second order ordinary differential equation which is called **"linear, with constant coefficients"**; it has the general form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

where a, b and c are the constant coefficients.

The various cases of solution which arise depend on the values of the coefficients, together with the type of function, $f(x)$, on the right hand side. These cases will now be dealt with in turn.

15.4.2 SECOND ORDER HOMOGENEOUS EQUATIONS

The term **"homogeneous"**, in the context of <u>second</u> order differential equations, is used to mean that the function, $f(x)$, on the right hand side is zero. It should not be confused with the previous use of this term in the context of <u>first</u> order differential equations.

We therefore consider equations of the general form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0.$$

Note:
A very simple case of this equation is

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0,$$

which, on integration twice, gives the general solution

$$y = Ax + B,$$

where A and B are arbitrary constants. We should therefore expect two arbitrary constants in the solution of any second order linear differential equation with constant coefficients.

The Standard General Solution

The equivalent of

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$$

in the discussion of first order differential equations would have been

$$b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0; \quad \text{that is,} \quad \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{c}{b}y = 0$$

and this could have been solved using an integrating factor of

$$e^{\int \frac{c}{b} \, \mathrm{d}x} = e^{\frac{c}{b}x},$$

giving the general solution

$$y = Ae^{-\frac{c}{b}x},$$

where A is an arbitrary constant.

It seems reasonable, therefore, to make a trial solution of the form $y = Ae^{mx}$, where $A \neq 0$, in the second order case.

We shall need

$$\frac{\mathrm{d}y}{\mathrm{d}x} = Ame^{mx} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = Am^2e^{mx}.$$

Hence, on substituting the trial solution, we require that

$$aAm^2e^{mx} + bAme^{mx} + cAe^{mx} = 0;$$

and, by cancelling Ae^{mx}, this condition reduces to

$$am^2 + bm + c = 0,$$

a quadratic equation, called the **"auxiliary equation"**, having the same (constant) coefficients as the orginal differential equation.

In general, it will have two solutions, say $m = m_1$ and $m = m_2$, giving corresponding solutions $y = Ae^{m_1 x}$ and $y = Be^{m_2 x}$ of the differential equation.

However, the linearity of the differential equation implies that the sum of any two solutions is also a solution, so that

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

is another solution; and, since this contains two arbitrary constants, we shall take it to be the general solution.

Notes:

(i) It may be shown that there are no solutions other than those of the above form though special cases are considered later.

(ii) It will be possible to determine particular values of A and B if an appropriate number of boundary conditions for the differential equation are specified. These will usually be a set of given values for y and $\frac{\mathrm{d}y}{\mathrm{d}x}$ at a certain value of x.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0$$

and also the particular solution for which $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -5$ when $x = 0$.

Solution

The auxiliary equation is $m^2 + 5m + 6 = 0$,

which can be factorised as

$$(m+2)(m+3)=0.$$

Its solutions are therefore $m=-2$ and $m=-3$.

Hence, the differential equation has general solution

$$y=Ae^{-2x}+Be^{-3x},$$

where A and B are arbitrary constants.

Applying the boundary conditions, we shall also need

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-2Ae^{-2x}-3Be^{-3x}.$$

Hence,

$$\begin{aligned} 2 &= A+B, \\ -5 &= -2A-3B \end{aligned}$$

giving $A=1$, $B=1$ and a particular solution

$$y=e^{-2x}+e^{-3x}.$$

15.4.3 SPECIAL CASES OF THE AUXILIARY EQUATION

(a) The auxilary equation has coincident solutions

Suppose that both solutions of the auxiliary equation are the same number, m_1.

In other words, the quadratic expression am^2+bm+c is a **"perfect square"**, which means that it is actually $a(m-m_1)^2$.

Apparently, the general solution of the differential equation is

$$y=Ae^{m_1x}+Be^{m_1x},$$

which does not genuinely contain <u>two</u> arbitrary constants since it can be rewritten as

$$y = Ce^{m_1 x} \quad \text{where} \quad C = A + B.$$

It will not, therefore, count as the <u>general</u> solution, though the fault seems to lie with the constants A and B rather than with m_1.

Consequently, let us now examine a new trial solution of the form

$$y = ze^{m_1 x},$$

where z denotes a function of x rather than a constant.

We shall also need

$$\frac{\mathrm{d}y}{\mathrm{d}x} = zm_1 e^{m_1 x} + e^{m_1 x}\frac{\mathrm{d}z}{\mathrm{d}x}$$

and

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = zm_1^2 e^{m_1 x} + 2m_1 e^{m_1 x}\frac{\mathrm{d}z}{\mathrm{d}x} + e^{m_1 x}\frac{\mathrm{d}^2 z}{\mathrm{d}x^2}.$$

On substituting these into the differential equation, we obtain the condition that

$$e^{m_1 x}\left[a\left(zm_1^2 + 2m_1\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{\mathrm{d}^2 z}{\mathrm{d}x^2}\right) + b\left(zm_1 + \frac{\mathrm{d}z}{\mathrm{d}x}\right) + cz\right] = 0$$

or

$$z(am_1^2 + bm_1 + c) + \frac{\mathrm{d}z}{\mathrm{d}x}(2am_1 + b) + a\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = 0.$$

The first term on the left hand side of this condition is zero since m_1 is already a solution of the auxiliary equation; and the second term is also zero since the auxiliary equation, $am^2 + bm + c = 0$, is equivalent to $a(m - m_1)^2 = 0$; that is, $am^2 - 2am_1 m + am_1^2 = 0$. Thus $b = -2am_1$.

We conclude that $\frac{d^2z}{dx^2} = 0$ with the result that $z = Ax + B$, where A and B are arbitrary constants.

The general solution of the differential equation in the case of coincident solutions to the auxiliary equation is therefore

$$y = (Ax + B)e^{m_1 x}.$$

EXAMPLE

Determine the general solution of the differential equation

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0.$$

Solution

The auxilary equation is

$$4m^2 + 4m + 1 = 0 \quad \text{or} \quad (2m + 1)^2 = 0$$

and it has coincident solutions at $m = -\frac{1}{2}$.

The general solution is therefore

$$y = (Ax + B)e^{-\frac{1}{2}x}.$$

(b) The auxiliary equation has complex solutions

If the auxiliary equation has complex solutions, they will automatically appear as a pair of **"complex conjugates"**, say $m = \alpha \pm j\beta$.

Using these two solutions instead of the previous m_1 and m_2, the general solution of the differential equation will be

$$y = Pe^{(\alpha + j\beta)x} + Qe^{(\alpha - j\beta)x},$$

where P and Q are arbitrary constants.

But, by properties of complex numbers, a neater form of this result is obtainable as follows:

$$y = e^{\alpha x}[P(\cos\beta x + j\sin\beta x) + Q(\cos\beta x - j\sin\beta x)]$$

or

$$y = e^{\alpha x}[(P+Q)\cos\beta x + j(P-Q)\sin\beta x].$$

Replacing $P+Q$ and $j(P-Q)$ (which are just arbitrary quantities) by A and B, we obtain the standard general solution for the case in which the auxiliary equation has complex solutions. It is

$$y = e^{\alpha x}[A\cos\beta x + B\sin\beta x].$$

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 13y = 0.$$

Solution

The auxiliary equation is

$$m^2 - 6m + 13 = 0,$$

which has solutions given by

$$m = \frac{-(-6) \pm \sqrt{(-6)^2 - 4\times 13\times 1}}{2\times 1} = \frac{6 \pm j4}{2} = 3 \pm j2.$$

The general solution is therefore

$$y = e^{3x}[A\cos 2x + B\sin 2x],$$

where A and B are arbitrary constants.

15.4.4 EXERCISES

1. Determine the general solutions of the following differential equations:

   (a)
   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 7\frac{\mathrm{d}y}{\mathrm{d}x} + 12y = 0;$$

   (b)
   $$\frac{\mathrm{d}^2r}{\mathrm{d}\theta^2} + 6\frac{\mathrm{d}r}{\mathrm{d}\theta} + 9r = 0;$$

   (c)
   $$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 4\frac{\mathrm{d}\theta}{\mathrm{d}t} + 5\theta = 0.$$

2. Solve the following differential equations, subject to the given boundary conditions:

   (a)
   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0,$$
   where $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$;

   (b)
   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 4\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = 0,$$
   where $x = 3$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 5$ when $t = 0$;

   (c)
   $$4\frac{\mathrm{d}^2z}{\mathrm{d}s^2} - 12\frac{\mathrm{d}z}{\mathrm{d}s} + 9z = 0,$$
   where $z = 1$ and $\frac{\mathrm{d}z}{\mathrm{d}s} = \frac{5}{2}$ when $s = 0$;

(d)

$$\frac{\mathrm{d}^2r}{\mathrm{d}\theta^2} - 2\frac{\mathrm{d}r}{\mathrm{d}\theta} + 2r = 0,$$

where $r = 5$ and $\frac{\mathrm{d}r}{\mathrm{d}\theta} = 7$ when $\theta = 0$.

15.4.5 ANSWERS TO EXERCISES

1. (a)

$$y = Ae^{-3x} + Be^{-4x};$$

(b)

$$r = (A\theta + B)e^{-3\theta};$$

(c)

$$\theta = e^{-2t}[A\cos 2t + B\sin 2t].$$

2. (a)

$$y = 3e^x - e^{2x};$$

(b)

$$x = 2e^t + e^{3t};$$

(c)

$$z = (s+1)e^{\frac{3}{2}s};$$

(d)

$$r = e^{\theta}[5\cos\theta + 2\sin\theta].$$