"JUST THE MATHS"

UNIT NUMBER

15.3

ORDINARY DIFFERENTIAL EQUATIONS 3 (First order equations (C))

by

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UNIT 15.3 - ORDINARY DIFFERENTIAL EQUATIONS 3

FIRST ORDER EQUATIONS (C)

15.3.1 LINEAR EQUATIONS

For certain kinds of first order differential equation, it is possible to multiply the equation throughout by a suitable factor which converts it into an exact differential equation.

For instance, the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y = x^2$$

may be multiplied throughout by x to give

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^3.$$

It may now be written

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x^3$$

and, hence, it has general solution

$$xy = \frac{x^4}{4} + C,$$

where C is an arbitrary constant.

Notes:

(i) The factor, x which has multiplied both sides of the differential equation serves as an **"integrating factor"**, but such factors cannot always be found by inspection.

(ii) In the discussion which follows, we shall develop a formula for determining integrating factors, in general, for what are known as **"linear differential equations"**.

DEFINITION

A differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

is said to be **"linear"**.

RESULT

Given the linear differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x),$$

the function

$$e^{\int P(x) \, \mathrm{d}x}$$

is always an integrating factor; and, on multiplying the differential equation throughout by this factor, its left hand side becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y \times e^{\int P(x) \, \mathrm{d}x}\right].$$

Proof

Suppose that the function, $R(x)$, is an integrating factor; then, in the equation

$$R(x)\frac{\mathrm{d}y}{\mathrm{d}x} + R(x)P(x)y = R(x)Q(x),$$

the left hand side must be the exact derivative of some function of x.

Using the formula for differentiating the product of two functions of x, we can **make** it the derivative of $R(x)y$ provided we can arrange that

$$R(x)P(x) = \frac{\mathrm{d}}{\mathrm{d}x}[R(x)].$$

But this requirement can be interpreted as a differential equation in which the variables $R(x)$ and x may be separated as follows:

$$\int \frac{1}{R(x)} \, \mathrm{d}R(x) = \int P(x) \, \mathrm{d}x.$$

Hence,

$$\ln R(x) = \int P(x) \, \mathrm{d}x.$$

That is,

$$R(x) = e^{\int P(x) \, \mathrm{d}x},$$

as required.

The solution is obtained by integrating the formula

$$\frac{\mathrm{d}}{\mathrm{d}x}[y \times R(x)] = R(x)P(x).$$

Note:
There is no need to include an arbitrary constant, C, when $P(x)$ is integrated, since it would only serve to introduce a constant factor of e^C in the above result, which would then immediately cancel out on multiplying the differential equation by $R(x)$.

EXAMPLES

1. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y = x^2.$$

Solution

An integrating factor is

$$e^{\int \frac{1}{x} \, \mathrm{d}x} = e^{\ln x} = x.$$

On multiplying throughout by the integrating factor, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}[y \times x] = x^3;$$

and so,

$$yx = \frac{x^4}{4} + C,$$

where C is an arbitrary constant.

2. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 2e^{-x^2}.$$

Solution
An integrating factor is

$$e^{\int 2x \ \mathrm{d}x} = e^{x^2}.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y \times e^{x^2}\right] = 2,$$

giving

$$ye^{x^2} = 2x + C,$$

where C is an arbitrary constant.

15.3.2 BERNOUILLI'S EQUATION

A similar type of differential equation to that in the previous section has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n.$$

It is called **"Bernouilli's Equation"** and may be converted to a linear differential equation by making the substitution

$$z = y^{1-n}.$$

Proof

The differential equation may be rewritten as

$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y^{1-n} = Q(x).$$

Also,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Hence the differential equation becomes

$$\frac{1}{1-n}\frac{\mathrm{d}z}{\mathrm{d}x} + P(x)z = Q(x).$$

That is,

$$\frac{\mathrm{d}z}{\mathrm{d}x} + (1-n)P(x)z = (1-n)Q(x),$$

which is a linear differential equation.

Note:
It is better not to regard this as a standard formula, but to apply the method of obtaining it in the case of particular examples.

EXAMPLES

1. Determine the general solution of the differential equation

$$xy - \frac{\mathrm{d}y}{\mathrm{d}x} = y^3 e^{-x^2}.$$

Solution
The differential equation may be rewritten

$$-y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} + x.y^{-2} = e^{-x^2}.$$

Substituting $z = y^{-2}$, we obtain $\frac{\mathrm{d}z}{\mathrm{d}x} = -2y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x}$ and, hence,

$$\frac{1}{2}\frac{\mathrm{d}z}{\mathrm{d}x} + xz = e^{-x^2}$$

or

$$\frac{\mathrm{d}z}{\mathrm{d}x} + 2xz = 2e^{-x^2}.$$

An integrating factor for this equation is

$$e^{\int \ 2x \ \mathrm{d}x} = e^{x^2}.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(ze^{x^2}\right) = 2,$$

giving

$$ze^{x^2} = 2x + C,$$

where C is an arbitrary constant.
Finally, replacing z by y^{-2},

$$y^2 = \frac{e^{x^2}}{2x + C}.$$

2. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = xy^2.$$

Solution

The differential equation may be rewritten

$$y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}.y^{-1} = x.$$

On substituting $z = y^{-1}$ we obtain $\frac{\mathrm{d}z}{\mathrm{d}x} = -y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x}$ so that

$$-\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{1}{x}.z = x$$

or

$$\frac{\mathrm{d}z}{\mathrm{d}x} - \frac{1}{x}.z = -x.$$

An integrating factor for this equation is

$$e^{\int \left(-\frac{1}{x}\right)\,\mathrm{d}x} = e^{-\ln x} = \frac{1}{x}.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(z \times \frac{1}{x}\right) = -1,$$

giving

$$\frac{z}{x} = -x + C,$$

where C is an arbitrary constant.

The general solution of the given differential equation is therefore

$$\frac{1}{xy} = -x + C \quad \text{or} \quad y = \frac{1}{Cx - x^2}.$$

15.3.3 EXERCISES

Use an integrating factor to solve the following differential equations subject to the given boundary condition:

1.

$$3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0,$$

where $y = 10$ when $x = 0$.

2.

$$3\frac{\mathrm{d}y}{\mathrm{d}x} - 5y = 10,$$

where $y = 4$ when $x = 0$.

3.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = 3x,$$

where $y = 2$ when $x = -1$.

4.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{1-x} = 1 - x^2,$$

where $y = 0$ when $x = -1$.

5.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\cot x = \cos x,$$

where $y = \frac{5}{2}$ when $x = \frac{\pi}{2}$.

6.

$$(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x} - xy = x,$$

where $y = 0$ when $x = 1$.

7.

$$3y - 2\frac{\mathrm{d}y}{\mathrm{d}x} = y^3 e^{4x},$$

where $y = 1$ when $x = 0$.

8.

$$2y - x\frac{\mathrm{d}y}{\mathrm{d}x} = x(x-1)y^4,$$

where $y^3 = 14$ when $x = 1$.

15.3.4 ANSWERS TO EXERCISES

1.

$$y = 10e^{-\frac{2}{3}x}.$$

2.

$$y = 6e^{\frac{5}{3}x} - 2.$$

3.

$$yx = x^3 - 1.$$

4.

$$y = \frac{1}{2}(1-x)(1+x)^2.$$

5.

$$y = \frac{\sin x}{2} + \frac{2}{\sin x}.$$

6.

$$y = 1 + x^2 - \sqrt{2(1+x^2)}.$$

7.

$$y^2 = \frac{7e^{3x}}{e^{7x}+6}.$$

8.

$$y^3 = \frac{56x^6}{21x^6 - 24x^7 + 7}.$$