"JUST THE MATHS"

UNIT NUMBER

15.2

ORDINARY DIFFERENTIAL EQUATIONS 2 (First order equations (B))

by

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UNIT 15.2 - ORDINARY DIFFERENTIAL EQUATIONS 2

FIRST ORDER EQUATIONS (B)

15.2.1 HOMOGENEOUS EQUATIONS

A differential equation of the first order is said to be **"homogeneous"** if, on replacing x by λx and y by λy in all the parts of the equation except $\frac{dy}{dx}$, λ may be removed from the equation by cancelling a common factor of λ^n, for some integer n.

Note:
Some examples of homogeneous equations would be

$$(x+y)\frac{dy}{dx} + (4x - y) = 0, \quad \text{and} \quad 2xy\frac{dy}{dx} + (x^2 + y^2) = 0,$$

where, from the first of these, a factor of λ could be cancelled and, from the second, a factor of λ^2 could be cancelled.

15.2.2 THE STANDARD METHOD

It turns out that the substitution

$$\boxed{y = vx} \quad \left(\text{giving} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}\right),$$

always converts a homogeneous differential equation into one in which the variables can be separated. The method will be illustrated by examples.

EXAMPLES

1. Solve the differential equation
$$x\frac{dy}{dx} = x + 2y,$$
subject to the condition that $y = 6$ when $x = 6$.

   Solution
   If $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$, so that the differential equation becomes
$$x\left(v + x\frac{dy}{dx}\right) = x + 2vx.$$

That is,

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + 2v$$

or

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v.$$

On separating the variables,

$$\int \frac{1}{1+v}\ \mathrm{d}v = \int \frac{1}{x}\ \mathrm{d}x,$$

giving

$$\ln(1+v) = \ln x + \ln A,$$

where A is an arbitrary constant.
An alternative form of this solution, without logarithms, is

$$Ax = 1 + v$$

and, substituting back $v = \frac{y}{x}$, the solution becomes

$$Ax = 1 + \frac{y}{x}$$

or

$$y = Ax^2 - x.$$

Finally, if $y = 6$ when $x = 1$, we have $6 = A - 1$ and, hence, $A = 7$.
The required particular solution is thus

$$y = 7x^2 - x.$$

2. Determine the general solution of the differential equation

$$(x+y)\frac{\mathrm{d}y}{\mathrm{d}x} + (4x - y) = 0.$$

Solution

If $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$, so that the differential equation becomes

$$(x + vx)\left(v + x\frac{dv}{dx}\right) + (4x - vx) = 0.$$

That is,

$$(1 + v)\left(v + x\frac{dv}{dx}\right) + (4 - v) = 0$$

or

$$v + x\frac{dv}{dx} = \frac{v - 4}{v + 1}.$$

On further rearrangement, we obtain

$$x\frac{dv}{dx} = \frac{v - 4}{v + 1} - v = \frac{-4 - v^2}{v + 1};$$

and, on separating the variables,

$$\int \frac{v + 1}{4 + v^2}\, dv = -\int \frac{1}{x}\, dx$$

or

$$\frac{1}{2}\int \left[\frac{2v}{4 + v^2} + \frac{2}{4 + v^2}\right] dv = -\int \frac{1}{x}\, dx.$$

Hence,

$$\frac{1}{2}\left[\ln(4 + v^2) + \tan^{-1}\frac{v}{2}\right] = -\ln x + C,$$

where C is an arbitrary constant.

Substituting back $v = \frac{y}{x}$, gives the general solution

$$\frac{1}{2}\left[\ln\left(4 + \frac{y^2}{x^2}\right) + \tan^{-1}\left(\frac{y}{2x}\right)\right] = -\ln x + C.$$

3. Determine the general solution of the differential equation

$$2xy\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 + y^2) = 0.$$

Solution

If $y = vx$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$, so that the differential equation becomes

$$2vx^2\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (x^2 + v^2x^2) = 0.$$

That is,

$$2v\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (1 + v^2) = 0$$

or

$$2vx\frac{\mathrm{d}v}{\mathrm{d}x} = -(1 + 3v^2).$$

On separating the variables, we obtain

$$\int \frac{2v}{1 + 3v^2}\ \mathrm{d}x = -\int \frac{1}{x}\ \mathrm{d}x,$$

which gives

$$\frac{1}{3}\ln(1 + 3v^2) = -\ln x + \ln A,$$

where A is an arbitrary constant.
Hence,

$$(1 + 3v^2)^{\frac{1}{3}} = \frac{A}{x}$$

or, on substituting back $v = \frac{y}{x}$,

$$\left(\frac{x^2 + 3y^2}{x^2}\right)^{\frac{1}{3}} = Ax,$$

which can be written

$$x^2 + 3y^2 = Bx^5,$$

where $B = A^3$.

15.2.3 EXERCISES

Use the substitution $y = vx$ to solve the following differential equations subject to the given boundary condition:

1.

$$(2y - x)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + y,$$

where $y = 3$ when $x = -2$.

2.

$$(x^2 - y^2)\frac{\mathrm{d}y}{\mathrm{d}x} = xy,$$

where $y = 5$ when $x = 0$.

3.

$$x^3 + y^3 = 3xy^2\frac{\mathrm{d}y}{\mathrm{d}x},$$

where $y = 1$ when $x = 2$.

4.

$$x(x^2 + y^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 2y^3,$$

where $y = 2$ when $x = 1$.

5.

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - (y + \sqrt{x^2 - y^2}) = 0,$$

where $y = 0$ when $x = 1$.

15.2.4 ANSWERS TO EXERCISES

1.

$$y^2 - xy - x^2 = 11.$$

2.

$$y = 5e^{-\frac{x^2}{2y^2}}.$$

3.

$$x^3 - 2y^3 = 3x.$$

4.

$$3x^2y = 2(y^2 - x^2).$$

5.

$$e^{\sin^{-1}\frac{y}{x}} = x.$$