"JUST THE MATHS"

UNIT NUMBER

15.10

ORDINARY DIFFERENTIAL EQUATIONS 10 (Simultaneous equations (C))

by

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UNIT 15.10 - ORDINARY DIFFERENTIAL EQUATIONS 10

SIMULTANEOUS EQUATIONS (C)

15.10.1 MATRIX METHODS FOR NON-HOMOGENEOUS SYSTEMS

In Units 15.5, 15.6 and 15.7 , it was seen that, for a single linear differential equation with constant coefficients, the general solution is made up of a particular integral and a complementary function (the latter being the general solution of the corresponding homogeneous differential equation).

In the work which follows, a similar principle is applied to a pair of simultaneous non-homogeneous differential equations of the form

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= ax_1 + bx_2 + f(t), \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= cx_1 + dx_2 + g(t). \end{aligned}$$

The method will be illustrated by the following example, in which $f(t) \equiv 0$:

EXAMPLE

Determine the general solution of the simultaneous differential equations

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_2, - - - - - - - - - - - - -(1) \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -4x_1 - 5x_2 + g(t), - - - - -(2) \end{aligned}$$

where $g(t)$ is (a) t, (b) e^{2t} (c) $\sin t$, (d) e^{-t}.

Solutions

(i) First, we write the differential equations in matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g(t),$$

which may be interpreted as

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} + \mathrm{N}g(t) \quad \text{where} \quad \mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathrm{M} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \quad \text{and} \quad \mathrm{N} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(ii) Secondly, we consider the corresponding "homogeneous" system

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX},$$

for which the characteristic equation is

$$\begin{vmatrix} 0-\lambda & 1 \\ -4 & -5-\lambda \end{vmatrix} = 0,$$

and gives

$$\lambda(5+\lambda) + 4 = 0 \quad \text{or} \quad \lambda^2 + 5\lambda + 4 = 0 \quad \text{or} \quad (\lambda+1)(\lambda+4) = 0.$$

(iii) The eigenvectors of M are obtained from the homogeneous equations

$$\begin{aligned} -\lambda k_1 + k_2 &= 0, \\ -4k_1 - (5+\lambda)k_2 &= 0. \end{aligned}$$

Hence, in the case when $\lambda = -1$, we solve

$$\begin{aligned} k_1 + k_2 &= 0, \\ -4k_1 - 4k_2 &= 0, \end{aligned}$$

and these are satisfied by any two numbers in the ratio $k_1 : k_2 = 1 : -1$.

Also, when $\lambda = -4$, we solve

$$\begin{aligned} 4k_1 + k_2 &= 0, \\ -4k_1 - k_2 &= 0 \end{aligned}$$

which are satisfied by any two numbers in the ratio $k_1 : k_2 = 1 : -4$.

The complementary function may now be written in the form

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t},$$

where A and B are arbitrary constants.

(iv) In order to obtain a particular integral for the equation

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} + \mathrm{N}g(t),$$

we note the second term on the right hand side and investigate a trial solution of a similar form. The three cases in this example are as follows:

(a) $g(t) \equiv t$

$$\text{Trial solution } \mathrm{X} = \mathrm{P} + \mathrm{Q}t,$$

where P and Q are constant matrices of order 2×1.

We require that

$$\mathrm{Q} = \mathrm{M}(\mathrm{P} + \mathrm{Q}t) + \mathrm{N}t,$$

whereupon, equating the matrix coefficients of t and the constant matrices,

$$\mathrm{MQ} + \mathrm{N} = \mathbf{0} \text{ and } \mathrm{Q} = \mathrm{MP},$$

giving

$$Q = -M^{-1}N \ \text{ and } \ P = M^{-1}Q.$$

Thus, using

$$M^{-1} = \frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix},$$

we obtain

$$Q = -\frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix}$$

and

$$P = \frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.3125 \\ 0.25 \end{bmatrix}.$$

The general solution, in this case, is

$$X = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \begin{bmatrix} -0.3125 \\ 0.25 \end{bmatrix} + \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} t.$$

(b) $g(t) \equiv e^{2t}$

$$\text{Trial solution } \ X = Pe^{2t}$$

We require that

$$2Pe^{2t} = MPe^{2t} + Ne^{2t}.$$

That is,

$$2P = MP + N.$$

The matrix, P, may now be determined from the formula

$$(2\mathrm{I} - \mathrm{M})\mathrm{P} = \mathrm{N};$$

or, in more detail,

$$\begin{bmatrix} 2 & -1 \\ 4 & 7 \end{bmatrix}.\mathrm{P} = \mathrm{N}.$$

Hence,

$$\mathrm{P} = \frac{1}{18}\begin{bmatrix} 7 & 1 \\ -4 & 2 \end{bmatrix}.\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 7 \\ -4 \end{bmatrix}.$$

The general solution, in this case, is

$$\mathrm{X} = A\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix}e^{-4t} + \frac{1}{18}\begin{bmatrix} 7 \\ -4 \end{bmatrix}e^{2t}.$$

(c) $g(t) \equiv \sin t$

$$\text{Trial solution} \quad \mathrm{X} = \mathrm{P}\sin t + \mathrm{Q}\cos t.$$

We require that

$$\mathrm{P}\cos t - \mathrm{Q}\sin t = \mathrm{M}(\mathrm{P}\sin t + \mathrm{Q}\cos t) + \mathrm{N}\sin t.$$

Equating the matrix coefficients of $\cos t$ and $\sin t$,

$$\mathrm{P} = \mathrm{MQ} \quad \text{and} \quad -\mathrm{Q} = \mathrm{MP} + \mathrm{N},$$

which means that

$$-\mathrm{Q} = \mathrm{M}^2\mathrm{Q} + \mathrm{N} \quad \text{or} \quad (\mathrm{M}^2 + \mathrm{I})\mathrm{Q} = -\mathrm{N}.$$

Thus,

$$Q = -(M^2 + I)^{-1}N,$$

where

$$M^2 + I = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 20 & 22 \end{bmatrix}$$

and, hence,

$$Q = -\frac{1}{34} \begin{bmatrix} 22 & 5 \\ -20 & -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{34} \begin{bmatrix} -5 \\ 3 \end{bmatrix}.$$

Also,

$$P = MQ = \frac{1}{34} \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 3 \end{bmatrix} = \frac{1}{34} \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

The general solution, in this case, is

$$X = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \frac{1}{34} \begin{bmatrix} 3 \\ 5 \end{bmatrix} \sin t + \frac{1}{34} \begin{bmatrix} -5 \\ 3 \end{bmatrix} \cos t.$$

(d) $g(t) \equiv e^{-t}$

In this case, the function, $g(t)$, is already included in the complementary function and it becomes necessary to assume a particular integral of the form

$$X = (P + Qt)e^{-t},$$

where P and Q are constant matrices of order 2×1.

We require that

$$Qe^{-t} - (P + Qt)e^{-t} = M(P + Qt)e^{-t} + Ne^{-t},$$

whereupon, equating the matrix coefficients of te^{-t} and e^{-t}, we obtain

$$-\mathrm{Q} = \mathrm{MQ} \quad \text{and} \quad \mathrm{Q} - \mathrm{P} = \mathrm{MP} + \mathrm{N}.$$

The first of these conditions shows that Q is an eigenvector of the matrix M corresponding the eigenvalue -1 and so, from earlier work,

$$\mathrm{Q} = k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for any constant k.

Also,

$$(\mathrm{M} + \mathrm{I})\mathrm{P} = \mathrm{Q} - \mathrm{N};$$

or, in more detail,

$$\begin{bmatrix} 1 & 1 \\ -4 & -4 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = k\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence,

$$\begin{aligned} p_1 + p_2 &= k, \\ -4p_1 - 4p_2 &= -k - 1. \end{aligned}$$

Using $p_1 + p_2 = k$ and $p_1 + p_2 = \frac{k+1}{4}$, we deduce that $k = \frac{1}{3}$ and that the matrix P is given by

$$\mathrm{P} = \begin{bmatrix} l \\ \frac{1}{3} - l \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + l\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for any number, l.

Taking $l = 0$ for simplicity, a particular integral is therefore

$$\mathrm{X} = \frac{1}{3}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} t\right\} e^{-t}.$$

and the general solution is

$$\mathrm{X} = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \frac{1}{3}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} t\right\} e^{-t}.$$

Note:
In examples for which neither $f(t)$ nor $g(t)$ is identically equal to zero, the particular integral may be found by adding together the separate forms of particular integral for $f(t)$ and $g(t)$ and writing the system of differential equations in the form

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} + \mathrm{N}_1 f(t) + \mathrm{N}_2 g(t),$$

where

$$\mathrm{N}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \mathrm{N}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For instance, if $f(t) \equiv t$ and $g(t) \equiv e^{2t}$, the particular integral would take the form

$$\mathrm{X} = \mathrm{P} + \mathrm{Q}t + \mathrm{R}e^{2t},$$

where P, Q and R are matrices of order 2×1.

15.10.2 EXERCISES

1. Determine the general solutions of the following systems of simultaneous differential equations:

   (a)

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1 + 3x_2 + 5t, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 3x_1 + x_2 + e^{3t}. \end{aligned}$$

(b)

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 3x_1 + 2x_2 + t^2, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 4x_1 + x_2 + e^{-2t}.\end{aligned}$$

2. Determine the complete solutions of the following systems of differential equations, subject to the conditions given:

(a)

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= -x_1 + 9x_2 + 3, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 11x_1 + x_2 + e^{10t},\end{aligned}$$

given that $x_1 = \frac{1}{225}$ and $x_2 = -\frac{1}{100}$ when $t = 0$.

(b)

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 5x_1 + 2x_2 + 2t^2 + t, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -2x_1 + x_2,\end{aligned}$$

given that $x_1 = \frac{32}{27}$ and $x_2 = -\frac{12}{27}$ when $t = 0$.

(c)

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 8x_1 + x_2 + \sin t, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -5x_1 + 6x_2 + \cos t,\end{aligned}$$

given that $x_1 = 0$ and $x_2 = 0$ when $t = 0$.

15.10.3 ANSWERS TO EXERCISES

1. (a)

$$A\begin{bmatrix}1\\1\end{bmatrix}e^{4t}+B\begin{bmatrix}1\\-1\end{bmatrix}e^{-2t}+\frac{1}{32}\begin{bmatrix}-25\\15\end{bmatrix}+\frac{1}{8}\begin{bmatrix}5\\-15\end{bmatrix}t-\frac{1}{5}\begin{bmatrix}3\\2\end{bmatrix}e^{3t};$$

(b)

$$A\begin{bmatrix}1\\1\end{bmatrix}e^{5t}+B\begin{bmatrix}1\\-2\end{bmatrix}e^{-t}+\frac{2}{125}\begin{bmatrix}41\\-84\end{bmatrix}+\frac{2}{25}\begin{bmatrix}-9\\16\end{bmatrix}t+\frac{1}{5}\begin{bmatrix}1\\-4\end{bmatrix}t^2+\frac{1}{7}\begin{bmatrix}2\\-5\end{bmatrix}e^{-2t}.$$

2. (a)

$$-\frac{7}{45}\begin{bmatrix}1\\-1\end{bmatrix}e^{-10t}+\frac{13}{900}\begin{bmatrix}9\\11\end{bmatrix}e^{10t}+\frac{3}{100}\begin{bmatrix}1\\-11\end{bmatrix}+\frac{1}{180}\begin{bmatrix}0\\1\end{bmatrix}e^{10t}+\frac{1}{20}\begin{bmatrix}9\\11\end{bmatrix}te^{10t};$$

(b)

$$\left\{(2t+1)\begin{bmatrix}1\\-1\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\right\}e^{3t}+\frac{1}{27}\begin{bmatrix}5\\-12\end{bmatrix}+\frac{1}{27}\begin{bmatrix}1\\-22\end{bmatrix}t-\frac{2}{9}\begin{bmatrix}1\\2\end{bmatrix}t^2;$$

(c)

$$\frac{1}{145}\left\{e^{7t}\left(\begin{bmatrix}-1\\25\end{bmatrix}\cos 2t+\begin{bmatrix}-12\\-10\end{bmatrix}\sin 2t\right)+\begin{bmatrix}-17\\-10\end{bmatrix}\sin t+\begin{bmatrix}1\\-25\end{bmatrix}\cos t\right\}.$$