“JUST THE MATHS”

UNIT NUMBER

15.1

ORDINARY DIFFERENTIAL EQUATIONS 1 (First order equations (A))

by

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UNIT 15.1 - ORDINARY DIFFERENTIAL EQUATIONS 1

FIRST ORDER EQUATIONS (A)

15.1.1 INTRODUCTION AND DEFINITIONS

1. An **ordinary differential equation** is a relationship between an independent variable (such as x), a dependent variable (such as y) and one or more ordinary derivatives of y with respect to x.

   There is no discussion, in Units 15, of **partial** differential equations, which involve partial derivatives (see Units 14). Hence, in what follows, we shall refer simply to "differential equations".

   For example,

   $$\frac{\mathrm{d}y}{\mathrm{d}x} = xe^{-2x}, \quad x\frac{\mathrm{d}y}{\mathrm{d}x} = y, \quad x^2\frac{\mathrm{d}y}{\mathrm{d}x} + y\sin x = 0 \text{ and } \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{x-y}$$

   are differential equations.

2. The **"order"** of a differential equation is the order of the highest derivative which appears in it.

3. The **"general solution"** of a differential equation is the most general algebraic relationship between the dependent and independent variables which satisfies the differential equation.

   Such a solution will not contain any derivatives; but we shall see that it will contain one or more arbitrary constants (the number of these constants being equal to the order of the equation). The solution need not be an <u>explicit</u> formula for one of the variables in terms of the other.

4. A **"boundary condition"** is a numerical condition which must be obeyed by the solution. It usually amounts to the substitution of particular values of the dependent and independent variables into the general solution.

5. An **"initial condition"** is a boundary condition in which the independent variable takes the value zero.

6. A **"particular solution"** (or **"particular integral"**) is a solution which contains no arbitrary constants.

   Particular solutions are usually the result of appplying a boundary condition to a general solution.

15.1.2 EXACT EQUATIONS

The simplest kind of differential equation of the first order is one which has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x).$$

It is an elementary example of an **"exact differential equation"** because, to find its solution, all that it is necessary to do is integrate both sides with respect to x.

In other cases of exact differential equations, the terms which are not just functions of the independent variable only, need to be recognised as the exact derivative with respect to x of some known function (possibly involving both of the variables).

The method will be illustrated by examples.

EXAMPLES

1. Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x + 5,$$

subject to the boundary condition that $y = 2$ when $x = 1$.

Solution

By direct integration, the general solution is

$$y = x^3 - 3x^2 + 5x + C,$$

where C is an arbitrary constant.

From the boundary condition,

$$2 = 1 - 3 + 5 + C, \text{ so that } C = -1.$$

Thus the particular solution obeying the given boundary condition is

$$y = x^3 - 3x^2 + 5x - 1.$$

2. Solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^3,$$

subject to the boundary condition that $y = 4$ when $x = 2$.

Solution

The left hand side of the differential equation may be recognised as the exact derivative with respect to x of the function xy.

Hence, we may write

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x^3;$$

and, by direct integration, this gives

$$xy = \frac{x^4}{4} + C,$$

where C is an arbitrary constant.

That is,

$$y = \frac{x^3}{4} + \frac{C}{x}.$$

Applying the boundary condition,

$$4 = 2 + \frac{C}{2},$$

which implies that $C = 4$ and the particular solution is

$$y = \frac{x^3}{4} + \frac{4}{x}.$$

3. Determine the general solution to the differential equation

$$\sin x + \sin y + x\cos y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Solution

The second and third terms on the right hand side may be recognised as the exact derivative of the function $x \sin y$; and, hence, we may write

$$\sin x + \frac{\mathrm{d}}{\mathrm{d}x}(x \sin y) = 0.$$

By direct integration, we obtain

$$-\cos x + x \sin y = C,$$

where C is an arbitrary constant.

This result counts as the general solution without further modification; but an explicit formula for y in terms of x may, in this case, be written in the form

$$y = \mathrm{Sin}^{-1}\left[\frac{C + \cos x}{x}\right].$$

15.1.3 THE METHOD OF SEPARATION OF THE VARIABLES

The method of this section relates to differential equations of the first order which may be written in the form

$$P(y)\frac{\mathrm{d}y}{\mathrm{d}x} = Q(x).$$

Integrating both sides with respect to x gives

$$\int P(y)\frac{\mathrm{d}y}{\mathrm{d}x}\ \mathrm{d}x = \int Q(x)\ \mathrm{d}x.$$

But, from the formula for integration by substitution in Units 12.3 and 12.4, this simplifies to

$$\int P(y)\ \mathrm{d}y = \int Q(x)\ \mathrm{d}x.$$

Note:
The way to remember this result is to treat dx and dy, in the given differential equation, as if they were separate numbers; then rearrange the equation so that one side contains only y while the other side contains only x; that is, we **separate the variables**. The process is completed by putting an integral sign in front of each side.

EXAMPLES

1. Solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y,$$

subject to the boundary condition that $y = 6$ when $x = 2$.

Solution

The differential equation may be rearranged as

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x};$$

and, hence,

$$\int \frac{1}{y}\ \mathrm{d}y = \int \frac{1}{x}\ \mathrm{d}x,$$

giving

$$\ln y = \ln x + C.$$

Applying the boundary condition,

$$\ln 6 = \ln 2 + C,$$

so that

$$C = \ln 6 - \ln 2 = \ln\left(\frac{6}{2}\right) = \ln 3.$$

The particular solution is therefore

$$\ln y = \ln x + \ln 3 \quad \text{or} \quad y = 3x.$$

Note:

In a general solution where most of the terms are logarithms, the calculation can be made simpler by regarding the arbitrary constant itself as a logarithm, calling it $\ln A$, for instance, rather than C. In the above example, we would then write

$$\ln y = \ln x + \ln A \quad \text{simplifying} \quad \text{to} \quad y = Ax.$$

On applying the boundary condition, $6 = 2A$, so that $A = 3$ and the particular solution is the same as before.

2. Solve the differential equation

$$x(4-x)\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0,$$

subject to the boundary condition that $y = 7$ when $x = 2$.

Solution

The differential equation may be rearranged as

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x(4-x)}.$$

Hence,

$$\int \frac{1}{y}\ \mathrm{d}y = \int \frac{1}{x(4-x)}\ \mathrm{d}x;$$

or, using the theory of partial fractions,

$$\int \frac{1}{y}\ \mathrm{d}y = \int \left[\frac{\frac{1}{4}}{x} + \frac{\frac{1}{4}}{4-x}\right]\ \mathrm{d}x.$$

The general solution is therefore

$$\ln y = \frac{1}{4}\ln x - \frac{1}{4}\ln(4-x) + \ln A$$

or

$$y = A\left(\frac{x}{4-x}\right)^{\frac{1}{4}}.$$

Applying the boundary condition, $7 = A$, so that the particular solution is

$$y = 7\left(\frac{x}{4-x}\right)^{\frac{1}{4}}.$$

15.1.4 EXERCISES

1. Determine the general solution of the differential equation

$$\frac{dy}{dx} = x^5 + 3e^{-2x}.$$

2. Given that differential equation

$$x^2\frac{dy}{dx} + 2xy = \sin x$$

is exact, determine its general solution.

3. Given that the differential equation

$$\tan x\frac{dy}{dx} + y\sec^2 x = \cos 2x$$

is exact, determine the particular solution for which $y = 1$ when $x = \frac{\pi}{4}$.

4. Use the method of separation of the variables to determine the general solution of each of the following differential equations:

(a)

$$\frac{dx}{dy} = (x-1)(x+2);$$

(b)

$$x(y-3)\frac{dy}{dx} = 4y.$$

5. Use the method of separation of the variables to solve the following differential equations subject to the given boundary condition:

(a)

$$(1+x^3)\frac{dy}{dx} = x^2y,$$

where $y = 2$ when $x = 1$;

(b)

$$x^3 + (y+1)^2\frac{dy}{dx} = 0,$$

where $y = 0$ when $x = 0$.

15.1.5 ANSWERS TO EXERCISES

1.

$$y = \frac{x^6}{6} - \frac{3e^{-2x}}{2} + C.$$

2.

$$y = \frac{C - \cos x}{x^2}.$$

3.

$$y = \frac{3}{2}\cot x - \cos^2 x.$$

4. (a)

$$y = \ln\left[A\left(\frac{x-1}{x+2}\right)^{\frac{1}{3}}\right];$$

(b)

$$y = \ln[Ax^4y^3].$$

5. (a)

$$y^3 = 4(1 + x^3);$$

(b)

$$4[1 - (y+1)^3] = 3x^4.$$