"JUST THE MATHS"

UNIT NUMBER

14.9

PARTIAL DIFFERENTIATION 9
(Taylor's series)
for
(Functions of several variables)

by

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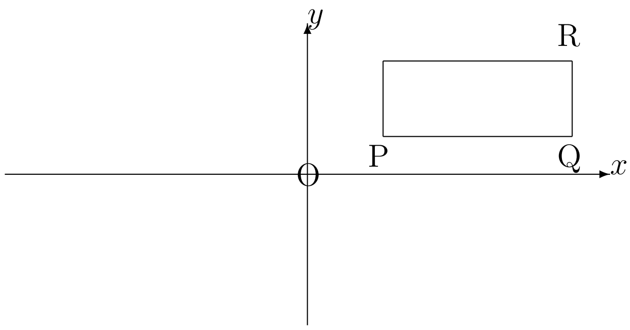
UNIT 14.9 - PARTIAL DIFFERENTIATION 9

TAYLOR'S SERIES FOR FUNCTIONS OF SEVERAL VARIABLES

14.9.1 THE THEORY AND FORMULA

Initially, we shall consider a function, $f(x, y)$, of **two** independent variables, x, y, and obtain a formula for $f(x + h, y + k)$ in terms of $f(x, y)$ and its partial derivatives.

Suppose that P,Q and R denote the points with cartesian co-ordinates, (x, y), $(x + h, y)$ and $(x + h, y + k)$, respectively.



(a) As we move in a straight line from P to Q, y remains constant so that $f(x, y)$ behaves as a function of x only.

Hence, by Taylor's theorem for one independent variable,

$$f(x + h, y) = f(x, y) + f_x(x, y) + \frac{h^2}{2!} f_{xx}(x, y) + \ldots,$$

where $f_x(x, y)$ and $f_{xx}(x, y)$ mean $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$ respectively, with similar notations encountered in what follows.

In abbreviated notation,

$$f(\text{Q}) = f(\text{P}) + hf_x(\text{P}) + \frac{h^2}{2!} f_{xx}(\text{P}) + \ \ldots$$

(b) As we move in a straight line from Q to R, x remains constant so that $f(x, y)$ behaves as a function of y only.

Hence,

$$f(x+h, y+k) = f(x+h, y) + kf_x(x+h, y) + \frac{k^2}{2!}f_{xx}(x+h, y) + \ldots;$$

or, in abbreviated notation,

$$f(\mathrm{R}) = f(\mathrm{Q}) + kf_y(\mathrm{Q}) + \frac{k^2}{2!}f_{yy}(\mathrm{Q}) + ..$$

(c) From the result in (a)

$$f_y(\mathrm{Q}) = f_y(\mathrm{P}) + hf_{yx}(\mathrm{P}) + \frac{h^2}{2!}f_{yxx}(\mathrm{P}) + \ldots$$

and

$$f_{yy}(\mathrm{Q}) = f_{yy}(\mathrm{P}) + hf_{yyx}(\mathrm{P}) + \frac{h^2}{2!}f_{yyxx}(\mathrm{Q}) + \ldots$$

(d) Substituting the results into (b) gives

$$f(\mathrm{R}) = f(\mathrm{P}) + hf_x(\mathrm{P}) + kf_y(\mathrm{P}) + \frac{1}{2!}\left[h^2 f_{xx}(\mathrm{P}) + 2hkf_{yx}(\mathrm{P}) + k^2 f_{yy}(\mathrm{P})\right] + ..$$

It may be shown that the complete result can be written as

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x, y) +$$

$$\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x, y) + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x, y) + \ \ldots$$

Notes:

(i) The equivalent of this result for a function of three variables would be

$$f(x+h,y+k,z+l) = f(x,y,z) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right) f(x,y,z) +$$

$$\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^2 f(x,y,z) + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^3 f(x,y,z) + \ \ldots$$

(ii) Alternative versions of Taylor's theorem may be obtained by interchanging $x, y, z...$ with $h, k, l....$

For example,

$$f(x+h,y+k) = f(h,k) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(h,k) +$$

$$\frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(h,k) + \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(h,k) + \ \ldots$$

(iii) Replacing x with $x-h$ and y with $y-k$ in (ii) gives the formula,

$$f(x,y) = f(h,k) + \left((x-h)\frac{\partial}{\partial x} + (y-k)\frac{\partial}{\partial y}\right) f(h,k) +$$

$$\frac{1}{2!}\left((x-h)\frac{\partial}{\partial x} + (y-k)\frac{\partial}{\partial y}\right)^2 f(h,k) + \frac{1}{3!}\left((x-h)\frac{\partial}{\partial x} + (y-k)\frac{\partial}{\partial y}\right)^3 f(h,k) + \ \ldots$$

This is called the **"Taylor expansion of $f(x,y)$ about the point (a,b)"**

(iv) A special case of Taylor's series (for two independent variables) is obtained by putting $h = 0$ and $k = 0$ in (ii) to give

$$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) + \ldots$$

This is called a **"MacLaurin's series"** but is also the Taylor expansion of $f(x,y)$ about the point $(0,0)$.

EXAMPLE

Determine the Taylor series expansion of the function $f\left(x+1, y+\frac{\pi}{3}\right)$ in ascending powers of x and y when

$$f(x,y) \equiv \sin xy,$$

neglecting terms of degree higher than two.

Solution

We use the result that

$$f\left(x+1, y+\frac{\pi}{3}\right) = f\left(1, \frac{\pi}{3}\right) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f\left(1, \frac{\pi}{3}\right) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f\left(1, \frac{\pi}{3}\right) + \ldots,$$

in which the first term on the right has value $\sqrt{3}/2$.

The partial derivatives required are as follows:

$$\frac{\partial f}{\partial x} \equiv y\cos xy \ \ \text{giving} \ \ -\frac{\pi}{6} \ \ \text{at} \ \ x = 1, \ \ y = \frac{\pi}{3};$$

$$\frac{\partial f}{\partial y} \equiv x\cos xy \ \ \text{giving} \ \ \frac{1}{2} \ \ \text{at} \ \ x = 1, \ \ y = \frac{\pi}{3};$$

$$\frac{\partial^2 f}{\partial x^2} \equiv -y^2\sin xy \ \ \text{giving} \ \ -\frac{\pi^2\sqrt{3}}{18} \ \ \text{at} \ \ x = 1, \ \ y = \frac{\pi}{3};$$

$$\frac{\partial^2 f}{\partial x \partial y} \equiv \cos xy - xy \sin xy \quad \text{giving} \quad \frac{1}{2} - \frac{\pi\sqrt{3}}{6} \quad \text{at} \quad x = 1, \quad y = \frac{\pi}{3};$$

$$\frac{\partial^2 f}{\partial y^2} \equiv -x^2 \sin xy \quad \text{giving} \quad -\frac{\sqrt{3}}{2} \quad \text{at} \quad x = 1, \quad y = \frac{\pi}{3}.$$

Neglecting terms of degree higher than two, we have

$$\sin xy = \frac{\sqrt{3}}{2} + \frac{\pi}{6}x + \frac{1}{2}y - \frac{\sqrt{3}\pi^2}{36}x^2 + \left(\frac{1}{2} - \frac{\pi\sqrt{3}}{6}\right)xy - \frac{\sqrt{3}}{4}y^2 + \ldots$$

14.9.2 EXERCISES

1. If $f(x, y) \equiv x^3 - 3xy^2$, show that

$$f(2 + h, 1 + k) = 2 + 9h - 12k + 6\left(h^2 - hk - k^2\right) + h^3 - 3hk^2.$$

2. If $f(x, y) \equiv \sin x \cosh y$, evaluate all the partial derivatives of $f(x, y)$ up to order five at the point, $(x, y) = (0, 0)$, and, hence, show that

$$\sin x \cosh y = x - \frac{1}{6}\left(x^3 - 3xy^2\right) + \frac{1}{120}\left(x^5 - 10x^3y^2 + 5xy^4\right) + \ldots$$

3. If z is a function of two independent variables, x and y, where $y \equiv z - x \sin z$, evaluate all the partial derivatives of $z(x, y)$ up to order three at the point, $(x, y) = (0, 0)$, and, hence, show that

$$z(x, y) = y + xy + x^2y + \ldots$$