"JUST THE MATHS"

UNIT NUMBER

14.8

PARTIAL DIFFERENTIATION 8
(Dependent and independent functions)

by

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UNIT 14.8 - PARTIAL DIFFERENTIATION 8

DEPENDENT AND INDEPENDENT FUNCTIONS

14.8.1 THE JACOBIAN

Suppose that

$$u \equiv u(x, y) \quad \text{and} \quad v \equiv v(x, y)$$

are two functions of two independent variables, x and y; then, in general, it is not possible to express u solely in terms of v, nor v solely in terms of u.

However, on occasions, it may be possible, as the following illustrations demonstrate:

ILLUSTRATIONS

1. If

$$u \equiv \frac{x+y}{x} \quad \text{and} \quad v \equiv \frac{x-y}{y},$$

then

$$u \equiv 1 + \frac{x}{y} \quad \text{and} \quad v \equiv \frac{x}{y} - 1,$$

which gives

$$(u-1)(v+1) \equiv \frac{x}{y}.\frac{y}{x} \equiv 1.$$

Hence,

$$u \equiv 1 + \frac{1}{v+1} \quad \text{and} \quad v \equiv \frac{1}{u-1} - 1.$$

2. If

$$u \equiv x + y \quad \text{and} \quad v \equiv x^2 + 2xy + y^2,$$

then

$$v \equiv u^2 \quad \text{and} \quad u \equiv \pm\sqrt{v}.$$

If u and v are **not** connected by an identical relationship, they are said to be **"independent functions"**.

THEOREM

Two functions, $u(x,y)$ and $v(x,y)$, are independent if and only if

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \not\equiv 0.$$

Proof:

We prove an equivalent statement, namely that $u(x,y)$ and $v(x,y)$ are dependent if and only if

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \equiv 0.$$

(a) Suppose that v is dependent on u by virtue of the relationship

$$v \equiv v(u).$$

By expressing $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, we shall establish that the determinant, J, is identically equal to zero.

We have

$$\frac{\partial v}{\partial x} \equiv \frac{\mathrm{d}v}{\mathrm{d}u}.\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} \equiv \frac{\mathrm{d}v}{\mathrm{d}u}.\frac{\partial u}{\partial y}.$$

Thus,

$$\frac{\frac{\partial v}{\partial x}}{\frac{\partial u}{\partial x}} \equiv \frac{\frac{\partial v}{\partial y}}{\frac{\partial u}{\partial y}} \equiv \frac{\mathrm{d}v}{\mathrm{d}u}$$

or

$$\frac{\partial u}{\partial x}.\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}.\frac{\partial u}{\partial y} \equiv 0,$$

which means that the determinant, J, is identically equal to zero.

(b) Secondly, let us suppose that

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \equiv 0,$$

and attempt to prove that $u(x,y)$ and $v(x,y)$ are dependent.

In theory, we could express v in terms of u and x by eliminating y between $u(x,y)$ and $v(x,y)$.

We shall assume that

$$v \equiv A(u,x).$$

By expressing $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, we may show that $A(u,x)$ does not contain x.

We have

$$\left(\frac{\partial v}{\partial x}\right)_y \equiv \left(\frac{\partial A}{\partial u}\right)_x \cdot \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial A}{\partial x}\right)_u$$

and

$$\left(\frac{\partial v}{\partial y}\right)_x \equiv \left(\frac{\partial A}{\partial u}\right)_x \cdot \left(\frac{\partial u}{\partial y}\right)_x.$$

Hence, if the determinant, J, is identically equal to zero, we may say that

$$\begin{vmatrix} \left(\frac{\partial u}{\partial x}\right)_y & \left(\frac{\partial u}{\partial y}\right)_x \\ \left(\frac{\partial A}{\partial u}\right)_x \cdot \left(\frac{\partial u}{\partial y}\right)_y + \left(\frac{\partial A}{\partial x}\right)_u & \left(\frac{\partial A}{\partial u}\right)_x \cdot \left(\frac{\partial u}{\partial y}\right)_x \end{vmatrix} \equiv 0;$$

and, on expansion, this gives

$$\left(\frac{\partial u}{\partial y}\right)_x \cdot \left(\frac{\partial A}{\partial x}\right)_u \equiv 0.$$

If the first of these two is equal to zero, then u contains only x and, hence, x could be expressed in terms of u, giving v as a function of u only. If the second is equal to zero, then A contains no x's and, again, v is a function of u only.

Notes:

(i) The determinant

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

may also be denoted by

$$\frac{\partial(u,v)}{\partial(x,y)}$$

and is called the **"Jacobian determinant"** or simply the **"Jacobian"** of u and v with respect to x and y.

(ii) Similar Jacobian determinants may be used to test for the dependence or independence of three functions of three variables, four functions of four variables, and so on.

For example, the three functions

$$u \equiv u(x,y,z), \quad v \equiv v(x,y,z) \ \text{ and } \ w \equiv w(x,y,z)$$

are independent if and only if

$$J \equiv \frac{\partial(u,v,w)}{\partial(x,y,z)} \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \not\equiv 0.$$

ILLUSTRATIONS

1.

$$u \equiv \frac{x+y}{x} \quad \text{and} \quad v \equiv \frac{x-y}{y}$$

are **not** independent, since

$$J \equiv \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} \equiv \frac{1}{xy} - \frac{1}{xy} \equiv 0$$

2.

$$u \equiv x+y \quad \text{and} \quad v \equiv x^2+2xy+y^2$$

are **not** independent, since

$$J \equiv \begin{vmatrix} 1 & 1 \\ 2x+2y & 2x+2y \end{vmatrix} \equiv 0.$$

3.

$$u \equiv x^2+2y \quad \text{and} \quad v \equiv xy$$

are independent, since

$$J \equiv \begin{vmatrix} 2x & 2 \\ y & x \end{vmatrix} \equiv 2x^2 - 2y \not\equiv 0.$$

4.

$$u \equiv x^2-2y+z, \quad v \equiv x+3y^2-2z, \quad \text{and} \quad w \equiv 5x+y+z^2$$

are **not** independent, since

$$J \equiv \begin{vmatrix} 2x & -2 & 1 \\ 1 & 6y & -2 \\ 5 & 1 & 2z \end{vmatrix} \equiv 24xyz + 4x - 30y + 4z + 25 \not\equiv 0.$$

14.8.2 EXERCISES

1. Determine which of the following pairs of functions are independent:

   (a)
$$u \equiv x \cos y \quad \text{and} \quad v \equiv x \sin y;$$

   (b)
$$u \equiv x + y \quad \text{and} \quad v \equiv \frac{y}{x+y};$$

   (c)
$$u \equiv x - 2y \quad \text{and} \quad v \equiv x^2 + 4y^2 - 4xy + 3x - 6y;$$

   (d)
$$u \equiv x + 2y \quad \text{and} \quad v \equiv x^2 - y^2 + 2xy - x.$$

2. Show that
$$u \equiv x + y + z, \quad v \equiv x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$$
   and
$$w \equiv x^3 + y^3 + z^3 - 3xyz$$
   are dependent.
   Show also that w may be expressed as a linear combination of u^3 and uv.

3. Given that
$$x + y + z \equiv u, \quad y + z \equiv uv \quad \text{and} \quad z \equiv uvw,$$
   express x and y in terms of u, v and w.
   Hence, show that
$$\frac{\partial(x, y, z)}{\partial(u, v, w)} \equiv u^2 v.$$

14.8.3 ANSWERS TO EXERCISES

1. (a) Independent, since $J \equiv x$;
   (b) Independent, since $J \equiv 1/(x+y)$;
   (c) Dependent, since $J \equiv 0$;
   (d) Independent, since $J \equiv 2 - 2x - 6y$.
2. $$w \equiv \frac{1}{4}\left[u^3 + 3uv\right].$$
3. $$x \equiv u - uv \quad \text{and} \quad y \equiv uv - uvw.$$