"JUST THE MATHS"

UNIT NUMBER

14.7

PARTIAL DIFFERENTIATION 7
(Change of independent variable)

by

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UNIT 14.7 - PARTIAL DIFFERENTIATON 7

CHANGE OF INDEPENDENT VARIABLE

14.7.1 ILLUSTRATIONS OF THE METHOD

In the theory of **"partial differential equations"** (that is, equations which involve partial derivatives) it is sometimes required to express a given equation in terms of a new set of independent variables. This would be necessary, for example, in changing a discussion from one geometrical reference system to another. The method is an application of the chain rule for partial derivatives and we illustrate it with examples.

EXAMPLES

1. Express, in plane polar co-ordinates, r and θ, the following partial differential equations:

   (a)
$$\frac{\partial V}{\partial x} + 5\frac{\partial V}{\partial y} = 1;$$

   (b)
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

   Solution

   Both differential equations involve a function, $V(x, y)$, where
$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

   Hence,
$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial V}{\partial y}.\frac{\partial y}{\partial r},$$

   or
$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}\cos\theta + \frac{\partial V}{\partial y}\sin\theta$$

   and
$$\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial V}{\partial y}.\frac{\partial y}{\partial \theta},$$

or

$$\frac{\partial V}{\partial \theta} = -\frac{\partial V}{\partial x} r \sin\theta + \frac{\partial V}{\partial y} r \cos\theta.$$

Now, we may eliminate, first $\frac{\partial V}{\partial y}$, and then $\frac{\partial V}{\partial x}$ to obtain

$$\frac{\partial V}{\partial x} = \cos\theta \frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}$$

and

$$\frac{\partial V}{\partial y} = \sin\theta \frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}.$$

Hence, differential equation, (a), becomes

$$(\cos\theta + 5\sin\theta)\frac{\partial V}{\partial r} + \left(\frac{5\cos\theta}{r} - \sin\theta\right)\frac{\partial V}{\partial \theta} = 1.$$

In order to find the <u>second</u>-order derivatives of V with respect to x and y, it is necessary to write the formulae for the first-order derivatives in the form

$$\frac{\partial}{\partial x}[V] = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)[V]$$

and

$$\frac{\partial}{\partial y}[V] = \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)[V].$$

From these, we obtain

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}\right),$$

which gives

$$\frac{\partial^2 V}{\partial x^2} = \cos^2\theta \frac{\partial^2 V}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial \theta} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r \partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial V}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2}.$$

Similarly,

$$\frac{\partial^2 V}{\partial y^2} = \sin^2\theta \frac{\partial^2 V}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial \theta} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r \partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial V}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2}.$$

Adding these together gives the differential equation, (b), in the form

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0.$$

2. Express the differential equation,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

(a) in cylindrical polar co-ordinates,

and

(b) in spherical polar co-ordinates.

Solution

(a) Using

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{and} \quad z = z,$$

we may use the results of the previous example to give

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

(b) Using

$$x = u\sin\phi\cos\theta, \quad y = u\sin\phi\sin\theta, \quad \text{and} \quad z = u\cos\phi,$$

we could write out three formulae for $\frac{\partial V}{\partial u}$, $\frac{\partial V}{\partial \theta}$ and $\frac{\partial V}{\partial \phi}$ and then solve for $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$; but this is complicated.

However, the result in part (a) provides a shorter method as follows:

Cylindrical polar co-ordinates are expressible in terms of spherical polar co-ordinates by the formulae

$$z = u\cos\phi, \quad r = u\sin\phi, \quad \theta = \theta.$$

Hence, by using the previous example with z, r, θ in place of x, y, z respectively and u, ϕ in place of r, θ, respectively, we obtain

$$\frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial r^2} = \frac{\partial^2 V}{\partial u^2} + \frac{1}{u}\frac{\partial V}{\partial u} + \frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2}.$$

Therefore, to complete the conversion we need only to consider $\frac{\partial V}{\partial r}$; and, by using r, u, ϕ in place of y, r, θ, respectively, the previous formula for $\frac{\partial V}{\partial y}$ gives

$$\frac{\partial V}{\partial r} = \sin\phi\frac{\partial V}{\partial u} + \frac{\cos\phi}{u}\frac{\partial V}{\partial \phi}.$$

The given differential equation thus becomes

$$\frac{\partial^2 V}{\partial u^2} + \frac{1}{u}\frac{\partial V}{\partial u} + \frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{1}{u^2 \sin^2\phi}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{u\sin\phi}\left[\sin\phi\frac{\partial V}{\partial u} + \frac{\cos\phi}{u}\frac{\partial V}{\partial \phi}\right] = 0.$$

That is,

$$\frac{\partial^2 V}{\partial u^2} + \frac{2}{u}\frac{\partial V}{\partial u} + \frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\cot\phi}{u^2}\frac{\partial V}{\partial \phi} + \frac{1}{u^2 \sin^2\phi}\frac{\partial^2 V}{\partial \theta^2} = 0.$$

14.7.2 EXERCISES

1. Express the partial differential equation,

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial x} = 0,$$

in plane polar co-ordinates, r and θ, where

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

2. Express the differential equation,

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = \frac{\partial V}{\partial z},$$

in spherical polar co-ordinates u, θ and ϕ, where

$$x = u\sin\phi\cos\theta, \quad y = u\sin\phi\sin\theta \quad \text{and} \quad z = u\cos\phi.$$

3. A function $\phi(x,t)$ satisfies the partial differential equation,

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{k^2}\frac{\partial^2\phi}{\partial t^2},$$

where k is a constant.

Express this equation in terms of new independent variables, u and v, where

$$x = \frac{1}{2}(u+v) \text{ and } t = \frac{1}{2k}(u-v).$$

4. A function $\theta(x,y)$ satisfies the partial differential equation,

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0.$$

Express this equation in terms of new independent variables, s and t, where

$$x = \ln u \text{ and } y = \ln v.$$

Determine, also, an expression for $\frac{\partial^2\theta}{\partial x\partial y}$ in terms of θ, u and v.

14.7.3 ANSWERS TO EXERCISES

1.

$$\frac{\partial V}{\partial r} = 0.$$

2.

$$(\sin\phi - \cos\phi)\frac{\partial V}{\partial u} - \frac{\cos\phi + \sin\phi}{u}\frac{\partial V}{\partial \phi} = 0.$$

3.

$$\frac{\partial^2\phi}{\partial u\partial v} = 0.$$

4.

$$u^2\frac{\partial^2\theta}{\partial x^2} + v^2\frac{\partial^2\theta}{\partial v^2} + u\frac{\partial\theta}{\partial u} + v\frac{\partial\theta}{\partial v} = 0,$$

and

$$\frac{\partial^2\theta}{\partial x\partial y} = uv\frac{\partial^2\theta}{\partial u\partial v}.$$