“JUST THE MATHS”

UNIT NUMBER

14.6

PARTIAL DIFFERENTIATION 6
(Implicit functions)

by

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UNIT 14.6 - PARTIAL DIFFERENTIATION 6

IMPLICIT FUNCTIONS

14.6.1 FUNCTIONS OF TWO VARIABLES

The chain rule, encountered earlier, has a convenient application to implicit relationships of the form,

$$f(x, y) = \text{constant},$$

between two independent variables, x and y.

It provides a means of determining the total derivative of y with respect to x.

Explanation

Taking x as the single independent variable, we may interpret $f(x, y)$ as a function of x and y in which both x and y are functions of x.

Differentiating both sides of the relationship, $f(x, y) = \text{constant}$, with respect to x gives

$$\frac{\partial f}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

In other words,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

EXAMPLES

1. If
$$f(x,y) \equiv x^3 + 4x^2y - 3xy + y^2 = 0,$$
determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

   Solution
$$\frac{\partial f}{\partial x} = 3x^2 + 8xy - 3y \quad \text{and} \quad \frac{\partial f}{\partial y} = 4x^2 - 3x + 2y.$$
Hence,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3x^2 + 8xy - 3y}{4x^2 - 3x + 2y}.$$

2. If
$$f(x,y) \equiv x\sin(2x - 3y) + y\cos(2x - 3y),$$
determine and expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

   Solution
$$\frac{\partial f}{\partial x} = \sin(2x - 3y) + 2x\cos(2x - 3y) - 2y\sin(2x - 3y)$$
and
$$\frac{\partial f}{\partial y} = -3x\cos(2x - 3y) + \cos(2x - 3y) + 3y\sin(2x - 3y).$$
Hence,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin(2x - 3y) + 2x\cos(2x - 3y) - 2y\sin(2x - 3y)}{3x\cos(2x - 3y) - \cos(2x - 3y) - 3y\sin(2x - 3y)}.$$

14.6.2 FUNCTIONS OF THREE VARIABLES

For relationships of the form,

$$f(x,y,z) = \text{constant},$$

let us suppose that x and y are independent of each other.

Then, regarding $f(x,y,z)$ as a function of x, y and z, where x, y and z are **all** functions of x and y, the chain rule gives

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}.\frac{\partial z}{\partial x} = 0.$$

But,

$$\frac{\partial x}{\partial x} = 1 \text{ and } \frac{\partial y}{\partial x} = 0.$$

Hence,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}.\frac{\partial z}{\partial x} = 0,$$

giving

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}};$$

and, similarly,

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}.$$

EXAMPLES

1. If

$$f(x,y,z) \equiv z^2xy + zy^2x + x^2 + y^2 = 5,$$

determine expressions for $\frac{\mathrm{d}z}{\mathrm{d}x}$ and $\frac{\mathrm{d}z}{\mathrm{d}y}$.

Solution

$$\frac{\partial f}{\partial x} = z^2y + zy^2 + 2x,$$

$$\frac{\partial f}{\partial y} = z^2x + 2zyx + 2y$$

and

$$\frac{\partial f}{\partial z} = 2zxy + y^2x.$$

Hence,

$$\frac{\partial z}{\partial x} = -\frac{z^2y + zy^2 + 2x}{2zxy + y^2x}$$

and

$$\frac{\partial z}{\partial y} = -\frac{z^2x + 2zyx + 2y}{2zxy + y^2x}.$$

2. If

$$f(x, y, z) \equiv xe^{y^2+2z},$$

determine expressions for $\frac{\mathrm{d}z}{\mathrm{d}x}$ and $\frac{\mathrm{d}z}{\mathrm{d}y}$.

Solution

$$\frac{\partial f}{\partial x} = e^{y^2+2z},$$

$$\frac{\partial f}{\partial y} = 2yxe^{y^2+2z},$$

and

$$\frac{\partial f}{\partial z} = 2xe^{y^2+2z}.$$

Hence,

$$\frac{\partial z}{\partial x} = -\frac{e^{y^2+2z}}{2xe^{y^2+2z}} = -\frac{1}{2x}$$

and

$$\frac{\partial z}{\partial y} = -\frac{2yxe^{y^2+2z}}{2xe^{y^2+2z}} = -y.$$

14.6.3 EXERCISES

1. Use partial differentiation to determine expressions for $\frac{dy}{dx}$ in the following cases:

   (a)
   $$x^3 + y^3 - 2x^2y = 0;$$

   (b)
   $$e^x \cos y = e^y \sin x;$$

   (c)
   $$\sin^2 x - 5 \sin x \cos y + \tan y = 0.$$

2. If
   $$x^2y + y^2z + z^2x = 10,$$
   where x and y are independent, determine expressions for
   $$\frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}.$$

3. If
   $$xyz - 2\sin(x^2 + y + z) + \cos(xy + z^2) = 0,$$
   where x and y are independent, determine expressions for
   $$\frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}.$$

4. If
   $$r^2 \sin\theta = (r\cos\theta - 1)z,$$
   where r and θ are independent, determine expressions for
   $$\frac{\partial z}{\partial r} \text{ and } \frac{\partial z}{\partial \theta}.$$

14.6.4 ANSWERS TO EXERCISES

1. (a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4xy - 3x^2}{3y^2 - 2x^2};$$

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x \cos y - e^y \cos x}{x^x \sin y + e^y \sin x};$$

(c)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5\cos x \cos y - 2\sin x \cos x}{5\sin x siny + \sec^2 y}.$$

2.

$$\frac{\partial z}{\partial x} = -\frac{2xy + z^2}{y^2 + 2zx}$$

and

$$\frac{\partial z}{\partial y} = -\frac{x^2 + 2yz}{y^2 + 2zx}.$$

3.

$$\frac{\partial z}{\partial x} = -\frac{yz - 4x\cos(x^2 + y + z) - y\sin(xy + z^2)}{xy - 2\cos(x^2 + y + z) - 2z\sin(xy + z^2)}$$

and

$$\frac{\partial z}{\partial y} = -\frac{xz - 2\cos(x^2 + y + z) - x\sin(xy + z^2)}{xy - 2\cos(x^2 + y + z) - 2z\sin(xy + z^2)}.$$

4.

$$\frac{\partial z}{\partial r} = \frac{2r\sin\theta - z\cos\theta}{r\cos\theta - 1}$$

and

$$\frac{\partial z}{\partial \theta} = \frac{r^2\cos\theta + rz\sin\theta}{r\cos\theta - 1}.$$