"JUST THE MATHS"

UNIT NUMBER

14.5

PARTIAL DIFFERENTIATION 5
(Partial derivatives of composite functions)

by

A.J.Hobson

UNIT 14.5 - PARTIAL DIFFERENTIATION 5

PARTIAL DERIVATIVES OF COMPOSITE FUNCTIONS

14.5.1 SINGLE INDEPENDENT VARIABLES

In this Unit, we shall be concerned with functions, $f(x, y...)$, of two or more variables in which those variables are not independent, but are themselves dependent on some other variable, t.

The problem is to calculate the rate of increase (positive or negative) of such functions with respect to t.

Let us suppose that the variable, t, is subject to a small increment of δt, so that the variables $x, y...$ are subject to small increments of $\delta x,\ \delta y,\ \ldots$, respectively. Then the corresponding increment, δf, in $f(x, y...)$ is given by

$$\delta f \simeq \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \ldots,$$

where we note that no label other than f is being used, here, for the function of several variables. That is, it is not essential to use a specific **formula**, such as $w = f(x, y..)$.

Dividing throughout by δt gives

$$\frac{\delta f}{\delta t} \simeq \frac{\partial f}{\partial x}.\frac{\delta x}{\delta t} + \frac{\partial f}{\partial y}.\frac{\delta y}{\delta t} + \ldots$$

Allowing δt to tend to zero, we obtain the standard result for the **"total derivative"** of $f(x, y..)$ with respect to t, namely

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}t} + \ldots$$

This rule may be referred to as the **"chain rule"**, but more advanced versions of it will appear later.

EXAMPLES

1. A point, P, is moving along the curve of intersection of the surface whose cartesian equation is

$$\frac{x^2}{16} - \frac{y^2}{9} = z \quad \text{(a \quad Paraboloid)}$$

and the surface whose cartesian equation is

$$x^2 + y^2 = 5 \quad \text{(a \ Cylinder)}.$$

If x is increasing at 0.2 cms/sec, how fast is z changing when $x = 2$?

Solution

We may use the formula

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}t},$$

where

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.2 \ \text{ and } \ \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}.\frac{\mathrm{d}x}{\mathrm{d}t} = 0.2\frac{\mathrm{d}y}{\mathrm{d}x}.$$

But, from the equation of the paraboloid,

$$\frac{\partial z}{\partial x} = \frac{x}{8} \ \text{ and } \ \frac{\partial z}{\partial y} = -\frac{2y}{9};$$

and, from the equation of the cylinder,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$

Substituting $x = 2$ gives $y = \pm 1$ on the curve of intersection, so that

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \left(\frac{2}{8}\right)(0.2) + \left(-\frac{2}{9}\right)(\pm 1)(0.2)\left(\frac{-2}{\pm 1}\right) = 0.2\left(\frac{1}{4} + \frac{4}{9}\right) = \frac{5}{36} \text{ cms/sec.}$$

2. Determine the total derivative of u with respect to t in the case when

$$u = xy + yz + zx, \quad x = e^t, \quad y = e^{-t} \ \text{ and } \ z = x + y.$$

Solution

We may use the formula

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial u}{\partial z}.\frac{\mathrm{d}z}{\mathrm{d}t},$$

where

$$\frac{\partial u}{\partial x} = y + z, \quad \frac{\partial u}{\partial y} = z + x, \quad \frac{\partial u}{\partial z} = x + y$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}t} = e^t = x, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -e^{-t} = -y, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = e^t - e^{-t} = x - y.$$

Hence,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (y + z)x - (z + x)y + (x + y)(x - y)$$

$$= -zy + zx + x^2 - y^2$$

$$= z(x - y) + (x - y)(x + y).$$

That is,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (x - y)(x + y + z).$$

14.5.2 SEVERAL INDEPENDENT VARIABLES

We may now extend the work of the previous section to functions, $f(x, y..)$, of two or more variables in which $x, y..$ are each dependent on two or more variables, $s, t..$

Since the function, $f(x, y..)$, is dependent on $s, t..$, we may wish to determine its **partial** derivatives with respect to any one of these (independent) variables.

The result previously established for a **single** independent variable may easily be adapted as follows:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial s} + \ldots$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial t} + \ldots$$

Again, this is referred to as the **"chain rule"**.

EXAMPLES

1. Determine the first-order partial derivatives of z with respect to r and θ in the case when

$$z = x^2 + y^2, \quad \text{where} \quad x = r\cos\theta \quad \text{and} \quad y = r\sin 2\theta.$$

Solution

We may use the formulae

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial r}$$

and

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial \theta}.$$

These give

$$\text{(i)} \quad \frac{\partial z}{\partial r} = 2x\cos\theta + 2y\sin 2\theta$$

$$= 2r\left(\cos^2\theta + \sin^2 2\theta\right)$$

and

$$\text{(ii)}\quad \frac{\partial z}{\partial \theta} = 2x(-r\sin\theta) + 2y(2r\cos 2\theta)$$

$$= 2r^2\left(2\cos 2\theta\sin 2\theta - \cos\theta\sin\theta\right).$$

2. Determine the first-order partial derivatives of w with respect to u, θ and ϕ in the case when

$$w = x^2 + 2y^2 + 2z^2,$$

where

$$x = u\sin\phi\cos\theta, \quad y = u\sin\phi\sin\theta \;\text{ and }\; z = u\cos\phi.$$

Solution

We may use the formulae

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial u},$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial \theta}$$

and

$$\frac{\partial w}{\partial \phi} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial \phi} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial \phi} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial \phi}.$$

These give

$$\text{(i)}\quad \frac{\partial w}{\partial u} = 2x\sin\phi\cos\theta + 4y\sin\phi\sin\theta + 4z\cos\phi$$

$$= 2u\sin^2\phi\cos^2\theta + 4u\sin^2\phi\sin^2\theta + 4u\cos^2\phi;$$

$$\text{(ii)} \quad \frac{\partial w}{\partial \theta} = -2xu\sin\phi\sin\theta + 4yu\sin\phi\cos\theta$$

$$= -2u^2\sin^2\phi\sin\theta\cos\theta + 4u^2\sin^2\phi\sin\theta\cos\theta$$

$$= 2u^2\sin^2\phi\sin\theta\cos\theta;$$

$$\text{(iii)} \quad \frac{\partial w}{\partial \phi} = 2xu\cos\phi\cos\theta + 4yu\cos\phi\sin\theta - 4zu\sin\phi$$

$$= 2u^2\sin\phi\cos\phi\cos^2\theta + 4u^2\sin\phi\cos\phi\sin^2\theta - 4u^2\sin\phi\cos\phi$$

$$= 2u^2\sin\phi\cos\phi\left(\cos^2\theta + 2\sin^2\theta - 2\right).$$

14.5.3 EXERCISES

1. Determine the total derivative of z with respect to t in the cases when

   (a)

   $$z = x^2 + 3xy + 5y^2, \quad \text{where} \quad x = \sin t \quad \text{and} \quad y = \cos t;$$

   (b)

   $$z = \ln(x^2 + y^2), \quad \text{where} \quad x = e^{-t} \quad \text{and} \quad y = e^t;$$

   (c)

   $$z = x^2y^2 \quad \text{where} \quad x = 2t^3 \quad \text{and} \quad y = 3t^2.$$

2. If $z = f(x, y)$, show that, when y is a function of x,

   $$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}x}.$$

   Hence, determine $\frac{\mathrm{d}z}{\mathrm{d}x}$ in the case when $z = xy + x^2y$ and $y = \ln x$.

3. The base radius, r, of a cone is decreasing at a rate of 0.1cms/sec while the perpendicular height, h, is increasing at a rate of 0.2cms/sec. Determine the rate at which the volume, V, is changing when $r = 2$cm and $h = 3$cm. (**Hint:** $V = (\pi r^2 h)/3$).

4. A rectangular solid has sides of lengths 3cms, 4cms and 5cms. Determine the rate of increase of the length of the diagonal of the solid if the sides are increasing at rates of $\frac{1}{3}$cms./sec, $\frac{1}{4}$cms./sec and $\frac{1}{5}$cms/sec, respectively.

5. If
$$z = (2x + 3y)^2 \quad \text{where} \quad x = r^2 - s^2 \quad \text{and} \quad y = 2rs,$$
determine, in terms of r and s the first-order partial derivatives of z with respect to r and s.

6. If
$$z = f(x, y) \quad \text{where} \quad x = e^u \cos v \quad \text{and} \quad y = e^u \sin v,$$
show that
$$\frac{\partial z}{\partial u} = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \quad \text{and} \quad \frac{\partial z}{\partial v} = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}.$$

7. If
$$w = 5x - 3y^2 + 7z^3 \quad \text{where} \quad x = 2s + 3t, \quad y = s - t \quad \text{and} \quad z = 4s + t,$$
determine, in terms of s and t, the first order partial derivatives of w with respect to s and t.

14.5.4 ANSWERS TO EXERCISES

1. (a)
$$\frac{\mathrm{d}z}{\mathrm{d}t} = 3\cos 2t - 4\sin 2t;$$
(b)
$$\frac{\mathrm{d}z}{\mathrm{d}t} = 2\left[\frac{e^{4t} - 1}{e^{4t} + 1}\right];$$
(c)
$$\frac{\mathrm{d}z}{\mathrm{d}t} = 360t^9.$$

2.

$$\frac{\mathrm{d}z}{\mathrm{d}x} = y^2 + 2xy + 1 + x.$$

3. The volume is decreasing at a rate of approximately 0.42 cubic centimetres per second.
4. The diagonal is increasing at a rate of approximately 0.42 centimetres per second.
5.

$$\frac{\partial z}{\partial r} = 8(r^2 - s^2 + 3rs)(2r + 3s) \quad \text{and} \quad \frac{\partial z}{\partial s} = 8(r^2 - s^2 + 3rs)(3r - 2s).$$

6. Results follow immediately from the formulae

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial v}.$$

7.

$$\frac{\partial w}{\partial s} = 1344s^2 + 672st + 84t^2 - 6s + 6t + 10$$

and

$$\frac{\partial w}{\partial t} = 336s^2 + 168st + 21t^2 + 6s - 6t + 15.$$