"JUST THE MATHS"

UNIT NUMBER

14.4

PARTIAL DIFFERENTIATION 4
(Exact differentials)

by

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UNIT 14.4 - PARTIAL DIFFERENTIATION 4

EXACT DIFFERENTIALS

14.4.1 TOTAL DIFFERENTIALS

In Unit 14.3, use was made of expressions of the form,

$$\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \ldots,$$

as an approximation for the increment (or error), δf, in the function, $f(x, y, ...)$, when x, y etc. are subject to increments (or errors) of δx, δy etc., respectively.

The expression may be called the **"total differential"** of $f(x, y, ...)$ and may be denoted by $\mathrm{d}f$, giving

$$\mathrm{d}f \simeq \delta f.$$

OBSERVATIONS

Consider the formula,

$$\mathrm{d}f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \ \ldots$$

(a) In the special case when $f(x, y, ...) \equiv x$, we may conclude that $\mathrm{d}f = \delta x$ or, in other words,

$$\mathrm{d}x = \delta x.$$

(b) In the special case when $f(x, y, ...) \equiv y$, we may conclude that $\mathrm{d}f = \delta y$ or, in other words,

$$\mathrm{d}y = \delta y.$$

(c) Observations (a) and (b) imply that the total differential of each **independent** variable is the same as the small increment (or error) in that variable; but the total differential of the **dependent** variable is only <u>approximately</u> equal to the increment (or error) in that variable.

(d) All of the previous observations may be summarised by means of the formula

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \ldots$$

14.4.2 TESTING FOR EXACT DIFFERENTIALS

In general, an expression of the form

$$P(x, y, ...)\mathrm{d}x + Q(x, y, ...)\mathrm{d}y + \ldots$$

will not be the total differential of a function, $f(x, y, ...)$, unless the functions, $P(x, y, ...)$, $Q(x, y, ...)$ etc. can be identified with $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ etc., respectively.

If this **is** possible, then the expression is known as an **"exact differential"**.

RESULTS

(i) The expression

$$P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y$$

is an exact differential if and only if

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}.$$

Proof:

(a) If the expression,

$$P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y,$$

is an exact differential, $\mathrm{d}f$, then

$$\frac{\partial f}{\partial x} \equiv P(x, y) \text{ and } \frac{\partial f}{\partial y} \equiv Q(x, y).$$

Hence, it must be true that

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x} \left(\equiv \frac{\partial^2 f}{\partial x \partial y} \right).$$

(b) Conversely, suppose that

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}.$$

We can certainly say that

$$P(x,y) \equiv \frac{\partial u}{\partial x}$$

for some function $u(x,y)$, since $P(x,y)$ could be integrated partially with respect to x.

But then,

$$\frac{\partial Q}{\partial x} \equiv \frac{\partial P}{\partial y} \equiv \frac{\partial^2 u}{\partial y \partial x};$$

and, on integrating partially with respect to x, we obtain

$$Q(x,y) = \frac{\partial u}{\partial y} + A(y),$$

where $A(y)$ is an **arbitrary** function of y.

Thus,

$$P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = \frac{\partial u}{\partial x}\mathrm{d}x + \left(\frac{\partial u}{\partial y} + A(y) \right) \mathrm{d}y;$$

and the right-hand side is the exact differential of the function,

$$u(x,y) + \int A(y)\ \mathrm{d}y.$$

(ii) By similar reasoning, it may be shown that the expression

$$P(x,y,z)\mathrm{d}x + Q(x,y,z)\mathrm{d}y + R(x,y,z)\mathrm{d}z$$

is an exact differential, provided that

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \text{and} \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

ILLUSTRATIONS

1.

$$x\mathrm{d}x + y\mathrm{d}y = \mathrm{d}\left[\frac{1}{2}\left(x^2 + y^2\right)\right].$$

2.

$$y\mathrm{d}x + x\mathrm{d}y = \mathrm{d}[xy].$$

3.

$$y\mathrm{d}x - x\mathrm{d}y$$

is not an exact differential since

$$\frac{\partial y}{\partial y} = 1 \quad \text{and} \quad \frac{\partial(-x)}{\partial x} = -1.$$

4.

$$2\ln y\mathrm{d}x + (x+z)\mathrm{d}y + z^2\mathrm{d}z$$

is not an exact differential since

$$\frac{\partial(2\ln y)}{\partial y} = \frac{2}{y}, \quad \text{and} \quad \frac{\partial(x+z)}{\partial x} = 1.$$

14.4.3 INTEGRATION OF EXACT DIFFERENTIALS

In section 14.4.2, the second half of the proof of the condition for the expression,

$$P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y,$$

to be an exact differential suggests, also, a method of determining which function, $f(x, y)$, it is the total differential of. The method may be illustrated by the following examples:

EXAMPLES

1. Verify that the expression,

$$(x + y\cos x)\mathrm{d}x + (1 + \sin x)\mathrm{d}y,$$

is an exact differential, and obtain the function of which it is the total differential.

Solution

Firstly,

$$\frac{\partial}{\partial y}(x + y\cos x) \equiv \frac{\partial}{\partial x}(1 + \sin x) \equiv \cos x;$$

and, hence, the expression is an exact differential.

Secondly, suppose that the expression is the total differential of the function, $f(x, y)$.

Then,

$$\frac{\partial f}{\partial x} \equiv x + y\cos x \quad ---------- (1)$$

and

$$\frac{\partial f}{\partial y} \equiv 1 + \sin x. \quad -----------(2)$$

Integrating (1) partially with respect to x gives

$$f(x, y) \equiv \frac{x^2}{2} + y\sin x + A(y),$$

where $A(y)$ is an **arbitrary** function of y only.

Substituting this result into (2) gives

$$\sin x + \frac{\mathrm{d}A}{\mathrm{d}y} \equiv 1 + \sin x.$$

That is,

$$\frac{\mathrm{d}A}{\mathrm{d}y} \equiv 1;$$

and, hence,

$$A(y) \equiv y + \text{constant}.$$

We conclude that

$$f(x, y) \equiv \frac{x^2}{2} + y \sin x + y + \text{constant}.$$

2. Verify that the expression,

$$(yz + 2)\mathrm{d}x + (xz + 6y)\mathrm{d}y + (xy + 3z^2)\mathrm{d}z,$$

is an exact differential and obtain the function of which it is the total differential.

Solution

Firstly,

$$\frac{\partial}{\partial y}(yz + 2) \equiv \frac{\partial}{\partial x}(xz + 6y) \equiv z,$$

$$\frac{\partial}{\partial z}(xz + 6y) \equiv \frac{\partial}{\partial y}(xy + 3z^2) \equiv x,$$

and

$$\frac{\partial}{\partial x}(xy + 3z^2) \equiv \frac{\partial}{\partial z}(yz + 2) \equiv y,$$

so that the given expression is an exact differential.

Suppose it is the total differential of the function, $F(x, y, z)$.
Then,

$$\begin{aligned} \frac{\partial F}{\partial x} &\equiv yz + 2, \quad --------(1) \\ \frac{\partial F}{\partial y} &\equiv xz + 6y, \quad --------(2) \\ \frac{\partial F}{\partial z} &\equiv xy + 3z^2. \quad --------(3) \end{aligned}$$

Integrating (1) partially with respect to x gives

$$F(x, y, z) \equiv xyz + 2x + A(y, z),$$

where $A(y, z)$ is an arbitrary function of y and z only.
Substituting this result into both (2) and (3) gives

$$\begin{aligned} xz + \frac{\partial A}{\partial y} &\equiv xz + 6y, \\ xy + \frac{\partial A}{\partial z} &\equiv xy + 3z^2. \end{aligned}$$

That is,

$$\begin{aligned} \frac{\partial A}{\partial y} &\equiv 6y, \quad ------(4) \\ \frac{\partial A}{\partial z} &\equiv 3z^2. \quad ------(5) \end{aligned}$$

Integrating (4) partially with respect to y gives

$$A(y, z) \equiv 3y^2 + B(z),$$

where $B(z)$ is an arbitrary function of z only.
Substituting this result into (5) gives

$$\frac{\mathrm{d}B}{\mathrm{d}z} \equiv 3z^2,$$

which implies that

$$B(z) \equiv z^3 + \text{constant}.$$

We conclude that

$$F(x, y, z) \equiv xyz + 2x + 3y^2 + z^3 + \text{constant}.$$

14.4.4 EXERCISES

1. Verify which of the following are exact differentials and integrate those which are:

   (a)

   $$5x + 12y - 9)\mathrm{d}x + (2x + 5y - 4)\mathrm{d}y;$$

   (b)

   $$12x + 5y - 9)\mathrm{d}x + (5x + 2y - 4)\mathrm{d}y;$$

   (c)

   $$(3x^2 + 2y + 1)\mathrm{d}x + (2x + 6y^2 + 2)\mathrm{d}y;$$

   (d)

   $$(y - e^x)\mathrm{d}x + x\mathrm{d}y;$$

   (e)

   $$\frac{1}{x}\mathrm{d}x - \left(\frac{y}{x^2} + 2x\right)\mathrm{d}y;$$

   (f)

   $$\cos(x + y)\mathrm{d}x + \cos(y - x)\mathrm{d}y;$$

   (g)

   $$(1 - \cos 2x)\mathrm{d}y + 2y\sin 2x\mathrm{d}x.$$

2. Verify that the expression,

   $$3x^2\mathrm{d}x + 2yz\mathrm{d}y + y^2\mathrm{d}z,$$

   is an exact differential and obtain the function of which it is the total differential.

3. Verify that the expression,

$$e^{xy}[y \sin z \mathrm{d}x + x \sin z \mathrm{d}y + \cos z \mathrm{d}z],$$

is an exact differential and obtain the function of which it is the total differential.

14.4.5 ANSWERS TO EXERCISES

1. (a) Not exact;
   (b)
   $$6x^2 + 5xy - 9x + y^2 - 4y + \text{constant};$$
   (c)
   $$x^3 + 2xy + x + 2y^3 + 2y + \text{constant};$$
   (d)
   $$xy - e^x + \text{constant};$$
   (e) Not exact;
   (f) Not exact;
   (g)
   $$y(1 - \cos 2x) + \text{constant}.$$
2.
   $$x^3 + y^2 z;$$
3.
   $$e^{xy} \sin z.$$