"JUST THE MATHS"

UNIT NUMBER

14.3

PARTIAL DIFFERENTIATION 3
(Small increments and small errors)

by

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UNIT 14.3 - PARTIAL DIFFERENTIATION 3

SMALL INCREMENTS AND SMALL ERRORS

14.3.1 FUNCTIONS OF ONE INDEPENDENT VARIABLE - A RECAP

For functions of **one** independent variable, a discussion of small increments and small errors has already taken place in Unit 11.6.

It was established that, if a dependent variable, y, is related to an independent variable, x, by means of the formula

$$y = f(x),$$

then

(a) The **increment**, δy, in y, due to an increment of δx, in x is given (to the first order of approximation) by

$$\delta y \simeq \frac{\mathrm{d}y}{\mathrm{d}x}\delta x;$$

and, in much the same way,

(b) The **error**, δy, in y, due to an error of δx in x, is given (to the first order of approximation) by

$$\delta y \simeq \frac{\mathrm{d}y}{\mathrm{d}x}\delta x.$$

14.3.2 FUNCTIONS OF MORE THAN ONE INDEPENDENT VARIABLE

Let us consider, first, a function, z, of two independent variables, x and y, given by the formula

$$z = f(x, y).$$

If x is subject to a small increment (or a small error) of δx, while y remains constant, then the corrresponding increment (or error) of δz in z will be given approximately by

$$\delta z \simeq \frac{\partial z}{\partial x}\delta x.$$

Similarly, if y is subject to a small increment (or a small error) of δy, while x remains constant, then the corresponding increment (or error) of δz in z will be given approximately by

$$\delta z \simeq \frac{\partial z}{\partial y}\delta y.$$

It seems reasonable to assume, therefore, that, when x is subject to a small increment (or a small error) of δx **and** y is subject to a small increment (or a small error) of δy, then the corresponding increment (or error) of δz in z will be given approximately by

$$\delta z \simeq \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y.$$

It may be shown that, to the first order of approximation, this is indeed true.

Notes:

(i) To prove more rigorously that the above result is true, use would have to be made of the result known as **"Taylor's Theorem"** for a function of two independent variables.

In the present case, where $z = f(x, y)$, it would give

$$f(x+\delta x, y+\delta y) = f(x, y) + \left(\frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y\right) + \left(\frac{\partial^2 z}{\partial x^2}(\delta x)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\delta x \delta y + \frac{\partial^2 z}{\partial y^2}(\delta y)^2\right) + \ldots,$$

which shows that

$$\delta z = f(x + \delta x, y + \delta y) - f(x, y) \simeq \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

to the first order of approximation.

(ii) The formula for a function of two independent variables may be extended to functions of a greater number of independent variables by simply adding further appropriate terms to the right hand side.

For example, if

$$w = F(x, y, z),$$

then

$$\delta w \simeq \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z.$$

EXAMPLES

1. A rectangle has sides of length $x$cms. and $y$cms.

   Determine, approximately, in terms of x and y, the increment in the area, A, of the rectangle when x and y are subject to increments of δx and δy, respectively.

   Solution

   The area, A, is given by

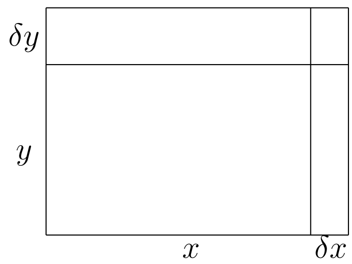
   $$A = xy,$$

   so that

   $$\delta A \simeq \frac{\partial A}{\partial x}\delta x + \frac{\partial A}{\partial y}\delta y = y\delta x + x\delta y.$$

   Note:

   The exact value of δA may be seen in the following diagram:



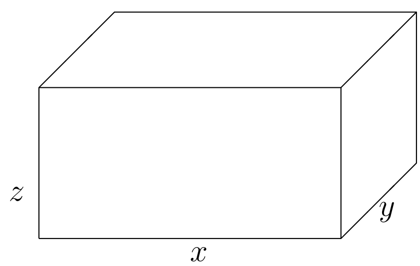
The difference between the approximate value and the exact value is represented by the area of the small rectangle having sides δxcms. and δycms.

2. In measuring a rectangular block of wood, the dimensions were found to be 10cms., 12cms and 20cms. with a possible error of $\pm 0.05 cms.$ in each.

   Calculate, approximately, the greatest possible error in the surface area, S, of the block and the percentage error so caused.

   Solution

   First, we may denote the lengths of the edges of the block by x, y and z.



The surface area, S, is given by

$$S = 2(xy + yz + zx),$$

which has the value 1120cms^2 when $x = 10$cms., $y = 12$cms. and $z = 20$cms.

Also,

$$\delta S \simeq \frac{\partial S}{\partial x}\delta x + \frac{\partial S}{\partial y}\delta y + \frac{\partial S}{\partial z}\delta z,$$

which gives

$$\delta S \simeq 2(y+z)\delta x + 2(x+z)\delta y + 2(y+x)\delta z;$$

and, on substituting $x = 10$, $y = 12$, $z = 20$, $\delta x = \pm 0.05$, $\delta y = \pm 0.05$ and $\delta z = \pm 0.05$, we obtain

$$\delta S \simeq \pm 2(12+20)(0.05) \pm 2(10+20)(0.05) \pm 2(12+10)(0.05).$$

The greatest error will occur when all the terms of the above expression have the same sign. Hence, the greatest error is given by

$$\delta S_{\max} \simeq \pm 8.4\text{cms.}^2;$$

and, since the originally calculated value was 1120, this represents a percentage error of approximately

$$\pm\frac{8.4}{1120} \times 100 = \pm 0.75$$

3. If

$$w = \frac{x^3 z}{y^4},$$

calculate, approximately, the percentage error in w when x is too small by 3%, y is too large by 1% and z is too large by 2%.

Solution

We have

$$\delta w \simeq \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z.$$

That is,

$$\delta w \simeq \frac{3x^2 z}{y^4}\delta x - \frac{4x^3 z}{y^5}\delta y + \frac{x^3}{y^4}\delta z,$$

where

$$\delta x = -\frac{3x}{100}, \quad \delta y = \frac{y}{100} \quad \text{and} \quad \delta z = \frac{2z}{100}.$$

Thus,

$$\delta w \simeq \frac{x^3 z}{y^4}\left[-\frac{9}{100} - \frac{4}{100} + \frac{2}{100}\right] = -\frac{11w}{100}.$$

The percentage error in w is given approximately by

$$\frac{\delta w}{w} \times 100 = -11.$$

That is, w is too small by approximately 11%.

14.3.3 THE LOGARITHMIC METHOD

In this section we consider again examples where it is required to calculate either a percentage increment or a percentage error.

We may conveniently use logarithms if the right hand side of the formula for the dependent variable involves a product, a quotient, or a combination of these two in which the independent variables are separated. This would be so, for instance, in the final example of the previous section.

The method is to take the natural logarithms of both sides of the equation before considering any partial derivatives; and we illustrate this, firstly, for a function of **two** independent variables.

Suppose that

$$z = f(x, y)$$

where $f(x, y)$ is the type of function described above.

Then,

$$\ln z = \ln f(x, y);$$

and, if we temporarily replace $\ln z$ by w, we have a new formula

$$w = \ln f(x, y).$$

The increment (or the error) in w, when x and y are subject to increments (or errors) of δx and δy respectively, is given by

$$\delta w \simeq \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y.$$

That is,

$$\delta w \simeq \frac{1}{f(x,y)}\frac{\partial f}{\partial x}\delta x + \frac{1}{f(x,y)}\frac{\partial f}{\partial y}\delta y = \frac{1}{f(x,y)}\left[\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y\right].$$

In other words,

$$\delta w \simeq \frac{1}{z}\left[\frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y\right].$$

We conclude that

$$\delta w \simeq \frac{\delta z}{z},$$

which means that the <u>fractional</u> increment (or error) in z approximates to the <u>actual</u> increment (or error) in $\ln z$. Multiplication by 100 will, of course, convert the fractional increment (or error) into a percentage.

Note:
The logarthmic method will apply equally well to a function of more than two independent variables where it takes the form of a product, a quotient, or a combination of these two.

EXAMPLES

1. If

$$w = \frac{x^3 z}{y^4},$$

calculate, approximately, the percentage error in w when x is too small by 3%, y is too large by 1% and z is too large by 2%.

Solution

Taking the natural logarithm of both sides of the given formula,

$$\ln w = 3\ln x + \ln z - 4\ln y,$$

giving

$$\frac{\delta w}{w} \simeq 3\frac{\delta x}{x} + \frac{\delta z}{z} - 4\frac{\delta y}{y},$$

where

$$\frac{\delta x}{x} = -\frac{3}{100}, \quad \frac{\delta y}{y} = \frac{1}{100} \quad \text{and} \quad \frac{\delta z}{z} = \frac{2}{100}.$$

Hence,

$$\frac{\delta w}{w} \times 100 = -9 + 2 - 4 = -13.$$

Thus, w is too small by approximately 11%, as before.

2. In the formula,

$$w = \sqrt{\frac{x^3}{y}},$$

x is subjected to an increase of 2%. Calculate, approximately, the percentage change needed in y to ensure that w remains unchanged.

Solution

Taking the natural logarithm of both sides of the formula,

$$\ln w = \frac{1}{2}[3\ln x - \ln y].$$

Hence,

$$\frac{\delta w}{w} \simeq \frac{1}{2}\left[3\frac{\delta x}{x} - \frac{\delta y}{y}\right],$$

where $\frac{\delta x}{x} = 0.02$, and we require that $\delta w = 0$.
Thus,

$$0 = \frac{1}{2}\left[0.06 - \frac{\delta y}{y}\right],$$

giving

$$\frac{\delta y}{y} = 0.06,$$

which means that y must be approximately 6% too large.

14.3.4 EXERCISES

1. A triangle is such that two of its sides (of length 6cms. and 8cms.) are at right-angles to each other.

   Calculate, approximately, the change in the length of the hypotenuse of the triangle when the shorter side is lengthened by 0.25cms. and the longer side is shortened by 0.125cms.

2. Two sides of a triangle are measured as $x = 150$cms. and $y = 200$cms. while the angle included between them is measured as $\theta = 60°$. Calculate the area of the triangle.

   If there are possible errors of ± 0.2cms. in the measurement of the sides and $\pm 1°$ in the angle, determine, approximately, the maximum possible error in the calculated area of the triangle.

   State your answers correct to the nearest whole number.

   (**Hint** use the formula, Area $= \frac{1}{2}xy\sin\theta$).

3. Given that the volume of a segment of a sphere is $\frac{1}{6}x(x^2 + 3y^2)$ where x is the height and y is the radius of the base, obtain, in terms of x and y, the percentage error in the volume when x is too large by 1% and y is too small by 0.5%.

4. If

   $$z = kx^{0.01}y^{0.08},$$

   where k is a constant, calculate, approximately, the percentage change in z when x is increased by 2% and y is decreased by 1%.

5. If

$$w = \frac{5xy^4}{z^3},$$

calculate, approximately, the maximum percentage error in w if x, y and z are subject to errors of $\pm 3\%$, $\pm 2.5\%$ and $\pm 4\%$, respectively.

6. If

$$w = 2xyz^{-\frac{1}{2}},$$

where x and z are subject to errors of 0.2%, calculate, approximately, the percentage error in y which results in w being without error.

14.3.5 ANSWERS TO EXERCISES

1. 0.05cms.
2. 12990cms.^{2} and 161cms.^{2}
3. $\frac{3x^2}{x^2+3y^2}$.
4. z decreases by 0.06%.
5. 25%.
6. -0.1%.