"JUST THE MATHS"

UNIT NUMBER

14.2

PARTIAL DIFFERENTIATION 2
(Partial derivatives of order higher than one)

by

A.J.Hobson

UNIT 14.2 - PARTIAL DIFFERENTIATION 2

PARTIAL DERIVATIVES OF THE SECOND AND HIGHER ORDERS

14.2.1 STANDARD NOTATIONS AND THEIR MEANINGS

In Unit 14.1, the partial derivatives encountered are known as partial derivatives of the **first order**; that is, the dependent variable was differentiated only **once** with respect to each independent variable.

But a partial derivative will, in general contain **all** of the independent variables, suggesting that we may need to differentiate again with respect to **any** of those variables.

For example, in the case where a variable, z, is a function of two independent variables, x and y, the possible partial derivatives of the second order are

(i)

$$\frac{\partial^2 z}{\partial x^2}, \quad \text{which means} \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right);$$

(ii)

$$\frac{\partial^2 z}{\partial y^2}, \quad \text{which means} \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right);$$

(iii)

$$\frac{\partial^2 z}{\partial x \partial y}, \quad \text{which means} \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right);$$

(iv)

$$\frac{\partial^2 z}{\partial y \partial x}, \quad \text{which means} \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right).$$

The last two can be shown to give the same result for all elementary functions likely to be encountered in science and engineering.

Note:

Occasionally, it may be necessary to use partial derivatives of order higher than two, as illustrated, for example, by

$$\frac{\partial^3 z}{\partial x \partial y^2}, \quad \text{which means} \quad \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right]$$

and

$$\frac{\partial^4 z}{\partial x^2 \partial y^2}, \quad \text{which means} \quad \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right]\right).$$

EXAMPLES

Determine all the first and second order partial derivatives of the following functions:

1.

$$z = 7x^3 - 5x^2y + 6y^3.$$

Solution

$$\frac{\partial z}{\partial x} = 21x^2 - 10xy; \qquad \frac{\partial z}{\partial y} = -5x^2 + 18y^2;$$

$$\frac{\partial^2 z}{\partial x^2} = 42x - 10y; \qquad \frac{\partial^2 z}{\partial y^2} = 36y;$$

$$\frac{\partial^2 z}{\partial y \partial x} = -10x; \qquad \frac{\partial^2 z}{\partial x \partial y} = -10x.$$

2.

$$z = y \sin x + x \cos y.$$

Solution

$$\frac{\partial z}{\partial x} = y \cos x + \cos y; \qquad \frac{\partial z}{\partial y} = \sin x - x \sin y;$$

$$\frac{\partial^2 z}{\partial x^2} = -y \sin x; \qquad \frac{\partial^2 z}{\partial y^2} = -x \cos y;$$

$$\frac{\partial^2 z}{\partial y \partial x} = \cos x - \sin y; \qquad \frac{\partial^2 z}{\partial x \partial y} = \cos x - \sin y.$$

3.

$$z = e^{xy}(2x - y).$$

Solution

$$\frac{\partial z}{\partial x} = e^{xy}[y(2x - y) + 2] = e^{xy}[2xy - y^2 + 2]; \qquad \frac{\partial z}{\partial y} = e^{xy}[x(2x - y) - 1] = e^{xy}[2x^2 - xy - 1];$$

$$\frac{\partial^2 z}{\partial x^2} = e^{xy}[y(2xy - y^2 + 2) + 2y] = e^{xy}[2xy^2 - y^3 + 4y]; \qquad \frac{\partial^2 z}{\partial y^2} = e^{xy}[x(2x^2 - xy - 1) - x] = e^{xy}[2x^3 - x^2y - 2x];$$

$$\frac{\partial^2 z}{\partial y \partial x} = e^{xy}[x(2xy - y^2 + 2) + 2x - 2y] = e^{xy}[2x^2y - xy^2 + 4x - 2y]; \qquad \frac{\partial^2 z}{\partial x \partial y} = e^{xy}[y(2x^2 - xy - 1) + 4x - y] = e^{xy}[2x^2y - xy^2 + 4x - 2y].$$

14.2.2 EXERCISES

1. Determine all the first and second order partial derivatives of the following functions:

   (a)

   $$z = 5x^2y^3 - 7x^3y^5;$$

   (b)

   $$z = x^4 \sin 3y.$$

2. Determine all the first and second order partial derivatives of the function

   $$w \equiv z^2 e^{xy} + x\cos(y^2 z).$$

3. If

   $$z = (x + y)\ln\left(\frac{x}{y}\right),$$

   show that

   $$x^2\frac{\partial^2 z}{\partial x^2} = y^2\frac{\partial^2 z}{\partial y^2}.$$

4. If

$$z = f(x + ay) + F(x - ay),$$

show that

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 z}{\partial y^2}.$$

14.2.3 ANSWERS TO EXERCISES

1. (a) The required partial derivatives are as follows:

$$\tfrac{\partial z}{\partial x} = 10xy^3 - 21x^2y^5; \qquad \tfrac{\partial z}{\partial y} = 15x^2y^2 - 35x^3y^4;$$

$$\tfrac{\partial^2 z}{\partial x^2} = 10y^3 - 42xy^5; \qquad \tfrac{\partial^2 z}{\partial y^2} = 30x^2y - 140x^3y^3;$$

$$\tfrac{\partial^2 z}{\partial y \partial x} = 30xy^2 - 105x^2y^4; \quad \tfrac{\partial^2 z}{\partial x \partial y} = 30xy^2 - 105x^2y^4.$$

(b) The required partial derivatives are as follows:

$$\tfrac{\partial z}{\partial x} = 4x^3 \sin 3y; \qquad \tfrac{\partial z}{\partial y} = 3x^4 \cos 3y;$$

$$\tfrac{\partial^2 z}{\partial x^2} = 12x^2 \sin 3y; \qquad \tfrac{\partial^2 z}{\partial y^2} = -9x^4 \sin 3y;$$

$$\tfrac{\partial^2 z}{\partial y \partial x} = 12x^3 \cos 3y; \qquad \tfrac{\partial^2 z}{\partial x \partial y} = 12x^3 \cos 3y.$$

2. The required partial derivatives are as follows:

$$\frac{\partial w}{\partial x} = yz^2e^{xy} + \cos(y^2z); \quad \frac{\partial w}{\partial y} = z^2xe^{xy} - 2xyz\sin(y^2z); \quad \frac{\partial w}{\partial z} = 2ze^{xy} - xy^2\sin(y^2z);$$

$$\frac{\partial^2 w}{\partial x^2} = y^2z^2e^{xy}; \quad \frac{\partial^2 w}{\partial y^2} = z^2x^2e^{xy} - 2xz\sin(y^2z) + 4xy^2z^2\cos(y^2z); \frac{\partial^2 w}{\partial z^2} = 2e^{xy} - xy^4\cos(y^2z);$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial x} = z^2 e^{xy} + z^2 xy e^{xy} - 2yz \sin(y^2 z);$$

$$\frac{\partial^2 w}{\partial y \partial z} = \frac{\partial^2 w}{\partial z \partial y} = 2zxe^{xy} - 2xy \sin(y^2 z) - 2xy^3 z \cos(y^2 z);$$

$$\frac{\partial^2 w}{\partial z \partial x} = \frac{\partial^2 w}{\partial x \partial z} = 2zye^{xy} - y^2 \sin(y^2 z).$$

3.

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{x} - \frac{y}{x^2} \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = -\frac{1}{y} + \frac{x}{y^2}.$$

4.

$$\frac{\partial^2 z}{\partial x^2} = f''(x + ay) + F''(x - ay) \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = a^2 f''(x + ay) + a^2 F''(x - ay).$$