"JUST THE MATHS"

UNIT NUMBER

14.11

PARTIAL DIFFERENTIATION 11
(Constrained maxima and minima)

by

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UNIT 14.11 - PARTIAL DIFFERENTIATION 11

CONSTRAINED MAXIMA AND MINIMA

Having discussed the determination of local maxima and local minima for a function, $f(x, y, ...)$, of several independent variables, we shall now consider that an additional constraint is imposed in the form of a relationship, $g(x, y, ...) = 0$.

This would occur, for example, if we wished to construct a container with the largest possible volume for a fixed value of the surface area.

14.11.1 THE SUBSTITUTION METHOD

The following examples illustrate a technique which may be used in elementary cases:

EXAMPLES

1. Determine any local maxima or local minima of the function,

$$f(x, y) \equiv 3x^2 + 2y^2,$$

subject to the constraint that $x + 2y - 1 = 0$.

Solution

In this kind of example, it is possible to eliminate either x or y by using the constraint. If we eliminate x, for instance, we may write $f(x, y)$ as a function, $F(y)$, of y only.

In fact,

$$f(x, y) \equiv F(y) \equiv 3(1 - 2y)^2 + 2y^2 \equiv 3 - 12y + 14y^2.$$

Using the principles of maxima and minima for functions of a single independent variable, we have

$$F'(y) \equiv 28y - 12 \quad \text{and} \quad F'' \equiv 28$$

and, hence, a local minimum occurs when $y = 3/7$ and hence, $x = 1/7$.

The corresponding local minimum value of $f(x, y)$ is

$$3\left(\frac{1}{7}\right)^2 + 2\left(\frac{3}{7}\right)^2 = \frac{21}{49} = \frac{3}{7}.$$

2. Determine any local maxima or local minima of the function,

$$f(x,y,z) \equiv x^2 + y^2 + z^2,$$

subject to the constraint that $x + 2y + 3z = 1$.

Solution

Eliminating x, we may write $f(x,y,z)$ as a function, $F(y,z)$, of y and z only.
In fact,

$$f(x,y,z) \equiv F(y,z) \equiv (1 - 2y - 3z)^2 + y^2 + z^2.$$

That is,

$$F(y,z) \equiv 1 - 4y - 6z + 12yz + 5y^2 + 10z^2.$$

Using the principles of maxima and minima for functions of two independent variables we have,

$$\frac{\partial F}{\partial y} \equiv -4 + 12z + 10y \ \text{ and } \ \frac{\partial F}{\partial z} \equiv -6 + 12y + 20z,$$

and a stationary value will occur when these are both equal to zero.
Thus,

$$\begin{aligned} 5y + 6z &= 2, \\ 6y + 10z &= 3, \end{aligned}$$

which give $y = 1/7$ and $z = 3/14$, on solving simultaneously.
The corresponding value of x is $1/14$, which gives a stationary value, for $f(x,y,z)$, of $14/(14)^2 = \frac{1}{14}$.
Also, we have

$$\frac{\partial^2 F}{\partial y^2} \equiv 10 > 0, \quad \frac{\partial^2 F}{\partial z^2} \equiv 20 > 0, \quad \text{and} \quad \frac{\partial^2 F}{\partial y \partial z} \equiv 12,$$

which means that

$$\frac{\partial^2 F}{\partial y^2} . \frac{\partial^2 F}{\partial z^2} - \left(\frac{\partial^2 F}{\partial y \partial z}\right)^2 = 200 - 144 > 0.$$

Hence there is a local minimum value, $\frac{1}{14}$, of $x^2 + y^2 + z^2$, subject to the constraint that $x + 2y + 3z = 1$, at the point where

$$x = \frac{1}{14}, \quad y = \frac{1}{7}, \quad \text{and} \quad z = \frac{3}{14}.$$

Note:
Geometrically, this example is calculating the square of the shortest distance from the origin onto the plane whose equation is $x + 2y + 3z = 1$.

14.11.2 THE METHOD OF LAGRANGE MULTIPLIERS

In determining the local maxima and local minima of a function, $f(x, y, ...)$, subject to the constraint that $g(x, y, ...) = 0$, it may be inconvenient (or even impossible) to eliminate one of the variables, $x, y, ...$

An alternative method may be illustrated by means of the following steps for a function of two independent variables:

(a) Suppose that the function, $z \equiv f(x, y)$, is subject to the constraint that $g(x, y) = 0$.

Then, since z is effectively a function of x only, its stationary values will be determined by the equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 0.$$

(b) From Unit 14.5 (Exercise 2), the total derivative of $z \equiv f(x, y)$ with respect to x, when x and y are not independent of each other, is given by the formula,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

(c) From the constraint that $g(x, y) = 0$, the process used in **(b)** gives

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and, hence, for all points on the surface with equation, $g(x, y) = 0$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}}.$$

Thus, throughout the surface with equation, $g(x, y) = 0$,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial f}{\partial x} - \left(\frac{\partial f}{\partial y}\right)\frac{\left(\frac{\partial g}{\partial x}\right)}{\left(\frac{\partial g}{\partial y}\right)}.$$

(d) Stationary values of z, subject to the constraint that $g(x, y) = 0$, will, therefore, occur when

$$\frac{\partial f}{\partial x}.\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}.\frac{\partial g}{\partial x} = 0.$$

But this may be interpreted as the condition that the two equations,

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} = 0,$$

should have a common solution for λ.

(e) Suppose that

$$\phi(x, y, \lambda) \equiv f(x, y) + \lambda g(x, y).$$

Then $\phi(x, y, \lambda)$ would have stationary values whenever its first order partial derivatives with respect to x, y and λ were equal to zero.

In other words,

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} = 0, \quad \frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} = 0, \quad \text{and} \quad g(x, y) = 0.$$

Conclusion

The stationary values of the function, $z \equiv f(x, y)$, subject to the constraint that $g(x, y) = 0$, occur at the points for which the function

$$\phi(x, y, \lambda) \equiv f(x, y) + \lambda g(x, y)$$

has stationary values.

The number, λ, is called a **"Lagrange multiplier"**.

Notes:

(i) In order to determine the nature of the stationary values of z, it will usually be necessary to examine the geometrical conditions in the neighbourhood of the stationary points.

(ii) The Lagrange multiplier method may also be applied to functions of three or more independent variables.

EXAMPLES

1. Determine any local maxima or local minima of the function,

   $$z \equiv 3x^2 + 2y^2,$$

   subject to the constraint that $x + 2y - 1 = 0$.

   Solution

   Firstly, we write

   $$\phi(x, y, \lambda) \equiv 3x^2 + 2y^2 + \lambda(x + 2y - 1).$$

   Then,

   $$\frac{\partial\phi}{\partial x} \equiv 6x + \lambda, \quad \frac{\partial\phi}{\partial y} \equiv 4y + 2\lambda \ \text{ and } \ \frac{\partial\phi}{\partial\lambda} \equiv x + 2y - 1.$$

   The third of these is already equal to zero; but we equate the first two to zero, giving

   $$\begin{aligned} 6x + \lambda &= 0, \\ 2y + \lambda &= 0. \end{aligned}$$

Eliminating λ shows that $6x - 2y = 0$, or $y = 3x$; and, if we substitute this into the constraint, we obtain $7x - 1 = 0$.

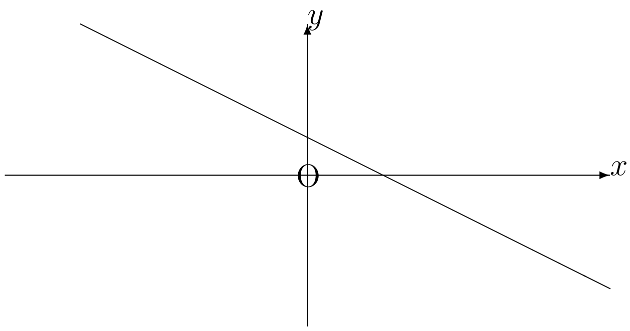
Hence,

$$x = \frac{1}{7}, \quad y = \frac{3}{7} \quad \text{and} \quad \lambda = -\frac{6}{7}.$$

A single stationary point therefore occurs at the point where

$$x = \frac{1}{7}, \quad y = \frac{3}{7} \quad \text{and} \quad z = 3\left(\frac{1}{7}\right)^2 + 2\left(\frac{3}{7}\right)^2 = \frac{3}{7}.$$

Finally, the geometrical conditions imply that the stationary value of z occurs at a point on the straight line whose equation is $x + 2y - 1 = 0$.



The stationary point is, in fact, a **minimum** value of z, since the function, $3x^2 + 2y^2$, has values larger than $3/7 \simeq 0.429$ at any point either side of the point, $(1/7, 3/7) = (0.14, 0.43)$, on the line whose equation is $x + 2y - 1 = 0$.

For example, at the points, $0.12, 0.44)$ and $(0.16, 0.42)$, on the line, the values of z are 0.4304 and 0.4296, respectively.

2. Determine the maximum and minimum values of the function, $z \equiv 3x + 4y$, subject to the constraint that $x^2 + y^2 = 1$.

   Solution

   Firstly, we write

$$\phi(x, y, \lambda) \equiv 3x + 4y + \lambda(x^2 + y^2 - 1).$$

Then,

$$\frac{\partial \phi}{\partial x} \equiv 3 + 2\lambda x, \quad \frac{\partial \phi}{\partial y} \equiv 4 + 2\lambda y \quad \text{and} \quad \frac{\partial \phi}{\partial \lambda} \equiv x^2 + y^2 - 1.$$

The third of these is already equal to zero; but we equate the first two to zero, giving

$$\begin{aligned} 3 + 2\lambda x &= 0, \\ 2 + \lambda y &= 0. \end{aligned}$$

Thus,

$$x = -\frac{3}{2\lambda} \quad \text{and} \quad y = \frac{2}{\lambda},$$

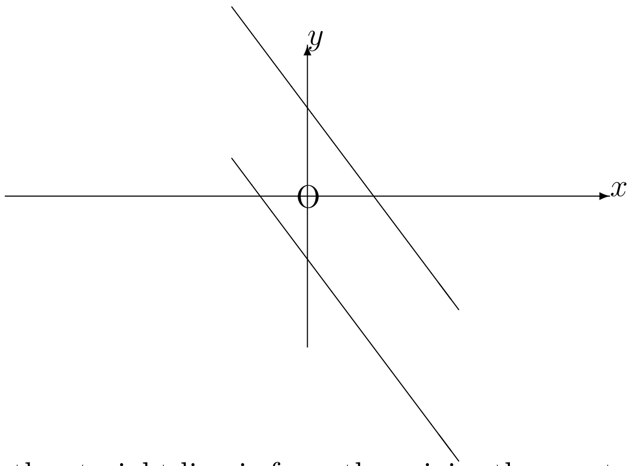
which we may substitute into the constraint to give

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1.$$

That is,

$$9 + 16 = 4\lambda^2 \quad \text{and} \quad \text{hence} \quad \lambda = \pm\frac{5}{2}.$$

We may deduce that $x = \pm\frac{3}{5}$ and $y = \pm\frac{4}{5}$, giving stationary values, ± 5, of z.
Finally, the geometrical conditions suggest that we consider a straight line with equation $3x + 4y = c$ (a constant) moving across the circle with equation $x^2 + y^2 = 1$.



The further the straight line is from the origin, the greater is the value of the constant, c.

The maximum and minimum values of $3x+4y$, subject to the constraint that $x^2+y^2=1$ will occur where the straight line touches the circle; and we have shown that these are the points, $(3/5, 4/5)$ and $(-3/5, -4/5)$.

3. Determine any local maxima or local minima of the function,

$$w \equiv x^2 + y^2 + z^2,$$

subject to the constraint that $x + 2y + 3z = 1$.

Solution

Firstly, we write

$$\phi(x, y, z, \lambda) \equiv x^2 + y^2 + z^2 + \lambda(x + 2y + 3z - 1).$$

Then,

$$\frac{\partial \phi}{\partial x} \equiv 2x + \lambda, \quad \frac{\partial \phi}{\partial y} \equiv 2y + 2\lambda, \quad \frac{\partial \phi}{\partial z} \equiv 2z + 3\lambda \;\text{ and }\; \frac{\partial \phi}{\partial \lambda} \equiv x + 2y + 3z - 1.$$

The fourth of these is already equal to zero; but we equate the first three to zero, giving

$$\begin{aligned} 2x + \lambda &= 0, \\ y + \lambda &= 0, \\ 2z + 3\lambda &= 0. \end{aligned}$$

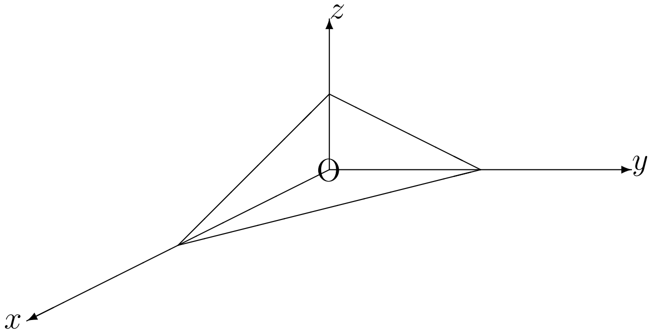
Eliminating λ shows that $2x - y = 0$, or $y = 2x$, and $6x - 2z = 0$, or $z = 3x$. Substituting these into the constraint gives $14x = 1$.
Hence,

$$x = \frac{1}{14}, \quad y = \frac{1}{7}, \quad z = \frac{3}{14} \text{ and } \lambda = -\frac{1}{7}.$$

A single stationary point therefore occurs at the point where

$$x = \frac{1}{14}, \quad y = \frac{1}{7}, \quad z = \frac{3}{14} \text{ and } w = \left(\frac{1}{14}\right)^2 + \left(\frac{1}{7}\right)^2 + \left(\frac{3}{14}\right)^2 = \frac{1}{14}.$$

Finally, the geometrical conditions imply that the stationary value of w occurs at a point on the plane whose equation is $x + 2y + 3z = 1$.



The stationary point must give a **minimum** value of w since the function, $x^2 + y^2 + z^2$, represents the square of the distance of a point, (x, y, z), from the origin; and, if the point is constrained to lie on a plane, this distance is bound to have a minimum value.

14.11.3 EXERCISES

1. In the following exercises, use both the substitution method and the Lagrange multiplier method:

   (a) Determine the minimum value of the function,

   $$z \equiv x^2 + y^2,$$

   subject to the constraint that $x + y = 1$.

   (b) Determine the maximum value of the function,

   $$z \equiv xy,$$

   subject to the constraint that $x + y = 15$.

(c) Determine the maximum value of the function,

$$z \equiv x^2 + 3xy - 5y^2,$$

subject to the constraint that $2x + 3y = 6$.

2. In the following exercises, use the Lagrange multiplier method:

   (a) Determine the maximum and minimum values of the function,

   $$w \equiv x - 2y + 5z,$$

   subject to the constraint that $x^2 + y^2 + z^2 = 30$.

   (b) If $x > 0$, $y > 0$ and $z > 0$, determine the maximum value of the function,

   $$w \equiv xyz,$$

   subject to the constraint that $x + y + z^2 = 16$.

   (c) Determine the maximum value of the function,

   $$w \equiv 8x^2 + 4yz - 16z + 600,$$

   subject to the constraint that $4x^2 + y^2 + 4z^2 = 16$.

14.11.4 ANSWERS TO EXERCISES

1. (a) The minimum value is $z = 1/2$, and occurs when $x = y = 1/2$;

   (b) The maximum value is $z \simeq 56.25$, and occurs when $x = y = 15/2$;

   (c) The maximum value is $z = 9$, and occurs when $x = 3$ and $y = 0$.

2. (a) The maximum value is 30, and occurs when $x = 1$, $y = -2$ and $z = 5$;
   The minimum value is -30, and occurs when $x = -1$, $y = 2$ and $z = -5$;

   (b) The maximum value is

   $$\frac{4096}{25\sqrt{5}} \simeq 73.27,$$

   and occurs when $x = 32/\sqrt{5}$, $y = 32/\sqrt{5}$ and $z = 4/\sqrt{5}$;

   (c) The maximum value is approximately 613.86, and occurs when $x = 0$, $y = -2$ and $z = \sqrt{3}$.