"JUST THE MATHS"

UNIT NUMBER

14.10

PARTIAL DIFFERENTIATION 10
(Stationary values)
for
(Functions of two variables)

by

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UNIT 14.10 - PARTIAL DIFFERENTIATION 10

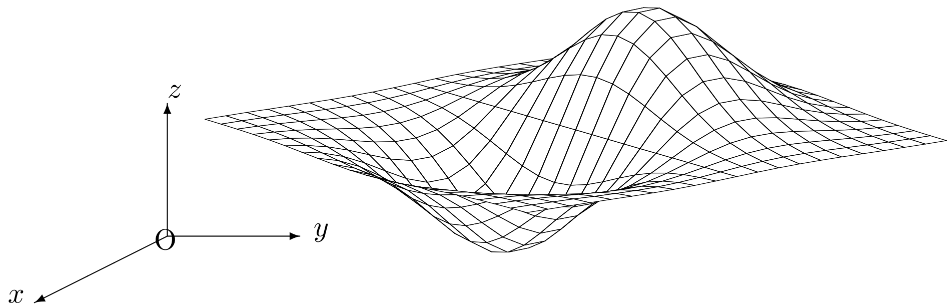
STATIONARY VALUES FOR FUNCTIONS OF TWO VARIABLES

14.10.1 INTRODUCTION

If $f(x, y)$ is a function of the two independent variables, x and y, then the equation,

$$z = f(x, y),$$

will normally represent some surface in space, referred to cartesian axes, Ox, Oy and Oz.



DEFINITION 1

The **"stationary points"**, on a surface whose equation is $z = f(x, y)$, are defined to be the points for which

$$\frac{\partial z}{\partial x} = 0 \text{ and } \frac{\partial z}{\partial y} = 0.$$

DEFINITION 2

The function, $z = f(x, y)$, is said to have a **"local maximum"** at a point, P(x_0, y_0, z_0), if z_0 is larger than the z co-ordinates of all other points on the surface, with equation $z = f(x, y)$, in the neighbourhood of P.

DEFINITION 3

The function $z = f(x, y)$ is said to have a **"local minimum"** at a point, P(x_0, y_0, z_0), if z_0 is smaller than the z co-ordinates of all other points on the surface, with equation $z = f(x, y)$, in the neighbourhood of P.

Note:
At a stationary point, $\mathrm{P}(x_0, y_0, z_0)$, on the surface with equation $z = f(x, y)$, each of the planes, $x = x_0$ and $y = y_0$, intersect the surface in a curve which has a stationary point at P.

14.10.2 SUFFICIENT CONDITIONS FOR MAXIMA AND MINIMA

A complete explanation of the conditions for a function, $z = f(x, y)$, to have a local maximum or a local minimum at a particular point require the use of Taylor's theorem for two variables.

At this stage, we state the standard set of sufficient conditions without proof.

(a) Sufficient conditions for a local maximum

A point, $\mathrm{P}(x_0, y_0, z_0)$, on the surface with equation $z = f(x, y)$, is a local maximum if

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0,$$

$$\frac{\partial^2 z}{\partial x^2} < 0 \ \text{ or } \ \frac{\partial^2 z}{\partial y^2} < 0$$

and

$$\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0,$$

when $x = x_0$ and $y = y_0$.

(b) Sufficient conditions for a local minimum

A point, $\mathrm{P}(x_0, y_0, z_0)$, on the surface with equation $z = f(x, y)$, is a local minimum if

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0,$$

$$\frac{\partial^2 z}{\partial x^2} > 0 \ \text{ or } \ \frac{\partial^2 z}{\partial y^2} > 0$$

and

$$\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0,$$

when $x = x_0$ and $y = y_0$.

Notes:

(i) If $\frac{\partial^2 z}{\partial x^2}$ is positive (or negative) and also $\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$, then $\frac{\partial^2 z}{\partial y^2}$ is automatically positive (or negative).

(ii) If it turns out that $\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$ is **negative** at P, we have what is called a **"saddle-point"**, irrespective of what $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ themselves are.

(iii) The values of z at the local maxima and local minima of the function, $z = f(x, y)$, may also be called the **"extreme values"** of the function, $f(x, y)$.

EXAMPLES

1. Determine the extreme values and the co-ordinates of any saddle-points of the function,

$$z = x^3 + x^2 - xy + y^2 + 4.$$

Solution

(i) First, we determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} = 3x^2 + 2x - y \quad \text{and} \quad \frac{\partial z}{\partial y} = -x + 2y.$$

(ii) Secondly, we solve the equations $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ for x and y.

$$\begin{aligned} 3x^2 + 2x - y &= 0, ----------(1) \\ -x + 2y &= 0. ---------- (2) \end{aligned}$$

Substituting equation (2) into equation (1) gives

$$3x^2 + 2x - \frac{1}{2}x = 0.$$

That is,

$$6x^2 + 3x = 0 \quad \text{or} \quad 3x(2x + 1) = 0.$$

Hence, $x = 0$ or $x = -\frac{1}{2}$, with corresponding values, $y = 0, \quad z = 4$ and $y = -\frac{1}{4}, \quad z = -\frac{65}{16}$, respectively.

The stationary points are thus $(0, 0, 4)$ and $\left(-\frac{1}{2}, -\frac{1}{4}, -\frac{65}{16}\right)$.

(iii) Thirdly, we evaluate $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$ at each stationary point.

$$\frac{\partial^2 z}{\partial x^2} = 6x + 2, \quad \frac{\partial^2 z}{\partial y^2} = 2, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = -1.$$

(a) At the point $(0, 0, 4)$,

$$\frac{\partial^2 z}{\partial x^2} = 2, \quad \frac{\partial^2 z}{\partial y^2} = 2, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = -1.$$

Hence,

$$\frac{\partial^2 z}{\partial x^2} > 0, \quad \frac{\partial^2 z}{\partial y^2} > 0, \quad \text{and} \quad \frac{\partial^2 z}{\partial x^2} . \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y} \right)^2 = 3 > 0$$

and, therefore, the point, $(0, 0, 4)$, is a local minimum, with z having a corresponding extreme value of 4.

(b) At the point $\left(-\frac{1}{2}, -\frac{1}{4}, \frac{65}{16}\right)$,

$$\frac{\partial^2 z}{\partial x^2} = -1, \quad \frac{\partial^2 z}{\partial y^2} = 2, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = -1.$$

Hence,

$$\frac{\partial^2 z}{\partial x^2} < 0, \quad \frac{\partial^2 z}{\partial y^2} > 0, \quad \text{and} \quad \frac{\partial^2 z}{\partial x^2} . \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y} \right)^2 = -3 < 0$$

and, therefore, the point, $\left(-\frac{1}{2}, -\frac{1}{4}, -\frac{65}{16}\right)$, is a saddle-point.

2. Determine the stationary points of the function,

$$z = 2x^3 + 6xy^2 - 3y^3 - 150x,$$

and determine their nature.

Solution

Following the same steps as in the previous example, we have

$$\frac{\partial z}{\partial x} = 6x^2 + 6y^2 - 150 \quad \text{and} \quad \frac{\partial z}{\partial y} = 12xy - 9y^2.$$

Hence, the stationary points occur where x and y are the solutions of the simultaneous equations,

$$\begin{aligned} x^2 + y^2 &= 25, - - - - - - - - - (1) \\ y(4x - 3y) &= 0. - - - - - - - - - - (2) \end{aligned}$$

From the second equation, $y = 0$ or $4x = 3y$.

Putting $y = 0$ in the first equation gives $x = \pm 5$ and, with these values of x and y, we obtain stationary points at $(5, 0, -500)$ and $(-5, 0, 500)$.

Putting $x = \frac{3}{4}y$ into the first equation gives $y = \pm 4$, $x = \pm 3$ and, with these values of x and y, we obtain stationary points at $(3, 4, -300)$ and $(-3, -4, 300)$.

To classify the stationary points we require

$$\frac{\partial^2 z}{\partial x^2} = 12x, \quad \frac{\partial^2 z}{\partial y^2} = 12x - 18y, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = 12y,$$

and the conclusions are given in the following table:

Point	$\frac{\partial^2 z}{\partial x^2}$	$\frac{\partial^2 z}{\partial y^2}$	$\frac{\partial^2 z}{\partial x \partial y}$	$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$	Nature
$(5, 0, -500)$	60	60	0	positive	minimum
$(-5, 0, 500)$	-60	-60	0	positive	maximum
$(3, 4, -300)$	36	-36	48	negative	saddle-point
$(-3, -4, 300)$	-36	36	-48	negative	saddle-point

Note:

The conditions used in the examples above are only **sufficient** conditions; that is, if the conditions are satisfied, we may make a conclusion. But it may be shown that there are stationary points which do **not** satisfy the conditions.

Outline proof of the sufficient conditions

From Taylor's theorem for two variables,

$$f(a+h,b+k)-f(a,b) = hf_x(a,b)+kf_y(a,b)+\frac{1}{2}\left(h^2 f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2 f_{yy}(a,b)\right)+\ldots,$$

where h and k are small compared with a and b, f_x means $\frac{\partial f}{\partial x}$, f_y means $\frac{\partial f}{\partial y}$, f_{xx} means $\frac{\partial^2 f}{\partial x^2}$, f_{yy} means $\frac{\partial^2 f}{\partial y^2}$ and f_{xy} means $\frac{\partial^2 f}{\partial x \partial y}$.

If $f_x(a,b) = 0$ and $f_y(a,b) = 0$, the conditions for a local minimum at the point $(a, b, f(a,b))$ will be satisfied when the second term on the right-hand side is positive; and the conditions for a local maximum at this point are satisfied when the second term on the right is negative.

We assume, here, that later terms of the Taylor series expansion are negligible.

Also, it may be shown that a quadratic expression of the form

$$Lh^2 + 2Mhk + Nk^2$$

is positive when $L > 0$ or $N > 0$ and $LN - M^2 > 0$; but negative when $L < 0$ or $N < 0$ and $LN - M^2 > 0$.

If it happens that $LN - M^2 < 0$, then it may be shown that the quadratic expression may take both positive and negative values.

Finally, replacing L, M and N by $f_{xx}(a,b)$, $f_{yy}(a,b)$ and $f_{xy}(a,b)$ respectively, the sufficient conditions for local maxima, local minima and saddle-points follow.

14.10.3 EXERCISES

1. Show that the function,

$$z = 3x^3 - y^3 - 4x + 3y,$$

has a local minimum value when $x = \frac{2}{3}$, $y = -1$ and calculate this minimum value. What other stationary points are there, and what is their nature ?

2. Determine the smallest value of the function,

$$z = 2x^2 + y^2 - 4x + 8y.$$

3. Show that the function,

$$z = 2x^2y^2 + x^2 + 4y^2 - 12xy,$$

has three stationary points and determine their nature.

4. Investigate the local extreme values of the function,

$$z = x^3 + y^3 + 9(x^2 + y^2) + 12xy.$$

5. Discuss the stationary points of the following functions and, where possible, determine their nature:

(a)

$$z = x^2 - 2xy + y^2;$$

(b)

$$z = xy.$$

Note:
It will not be possible to use all of the standard conditions; and a geometrical argument will be necessary.

14.10.4 ANSWERS TO EXERCISES

1. $\left(\frac{2}{3}, -1, -\frac{34}{9}\right)$ is a local minimum;
   $\left(-\frac{2}{3}, 1, \frac{34}{9}\right)$ is a local maximum;
   $\left(\frac{2}{3}, 1, \frac{2}{9}\right)$ is a saddle-point;
   $\left(-\frac{2}{3}, -1, -\frac{2}{9}\right)$ is a saddle-point.

2. The smallest value is -18, since there is a single local minimum at the point $(1, -4, -18)$.

3. $(0, 0, 0)$ is a saddle-point;
   $(2, 1, -8)$ is a local minimum; (**Hint:** try $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$)
   $(-2, -1, -8)$ is a local minimum.

4. $(0, 0, 0)$ is a local minimum;
   $(-10, -10, 1000)$ is a local maximum; (**Hint**: try $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}$)
   $(-4, 2, 28)$ is a saddle-point;
   $(2, -4, 28)$ is a saddle-point.

5. (a) Points $(\alpha, \alpha, 0)$ are such that $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$; but $\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$. In fact the surface is a "parabolic cylinder" which contains the straight line $x = y,\ z = 0$ and is symmetrical about the plane $x = y$.

   (b) $(0, 0, 0)$ is a saddle-point since z may have both positive and negative values in the neighbourhood of this point.