"JUST THE MATHS"

UNIT NUMBER

13.9

INTEGRATION APPLICATIONS 9
(First moments of a surface of revolution)

by

A.J.Hobson

UNIT 13.9 - INTEGRATION APPLICATIONS 9

FIRST MOMENTS OF A SURFACE OF REVOLUTION

13.9.1 INTRODUCTION

Suppose that C denotes an arc (with length s) in the xy-plane of cartesian co-ordinates, and suppose that δs is the length of a small element of this arc.

Then, for the surface obtained when the arc is rotated through 2π radians about the x-axis, the **"first moment"** about a plane through the origin, perpendicular to the x-axis, is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} 2\pi xy\delta s,$$

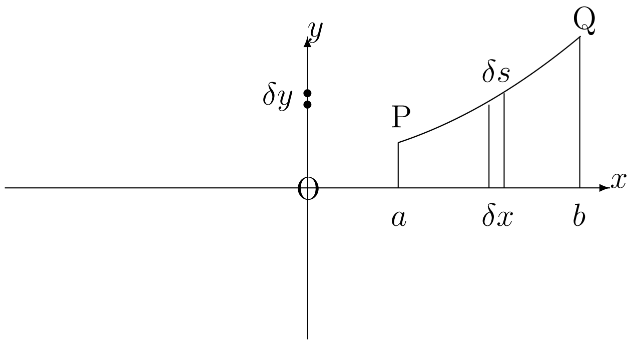
where x is the perpendicular distance, from the plane of moments, of the thin band, with surface area $2\pi y\delta s$, so generated.

13.9.2 INTEGRATION FORMULAE FOR FIRST MOMENTS

(a) Let us consider an arc of the curve whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may be divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

From Pythagoras' Theorem,

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x;$$

so that, for the surface of revolution of the arc about the x-axis, the first moment becomes

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi x y \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x = \int_a^b 2\pi x y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided that $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered. If not, then the above line needs to be prefixed by a negative sign.

From the technique of integration by substitution,

$$\int_a^b 2\pi xy\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x = \int_{t_1}^{t_2} 2\pi xy\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

We may conclude that the first moment about the plane through the origin, perpendicular to the x-axis is given by

$$\text{First Moment} = \pm\int_{t_1}^{t_2} 2\pi xy\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

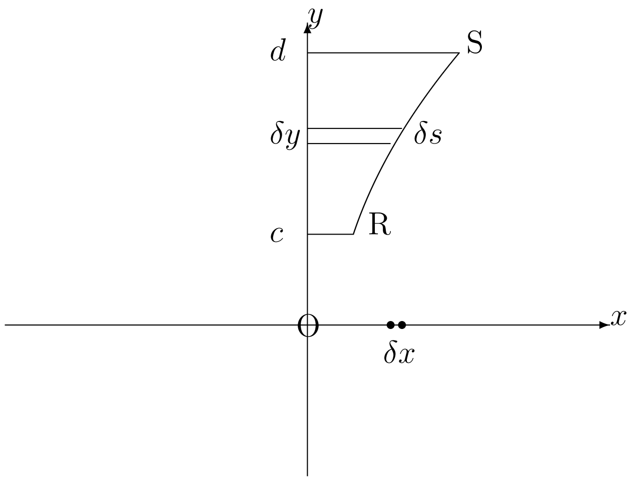
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in the previous section so that the first moment about a plane through the origin, perpendicular to the y-axis is given by

$$\int_c^d 2\pi yx\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$



Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$, then the first moment about a plane through the origin, perpendicular to the y-axis, is given by

$$\text{First moment} = \pm \int_{t_1}^{t_2} 2\pi yx \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative.

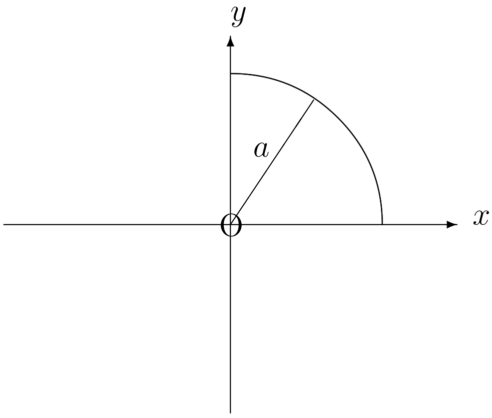
EXAMPLES

1. Determine the first moment about a plane through the origin, perpendicular to the x-axis, for the hemispherical surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



Using implicit differentiation, we have

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

The first moment about the specified plane is therefore given by

$$\int_0^a 2\pi xy\sqrt{1 + \frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a 2\pi xy\sqrt{\frac{x^2 + y^2}{y^2}}\ \mathrm{d}x.$$

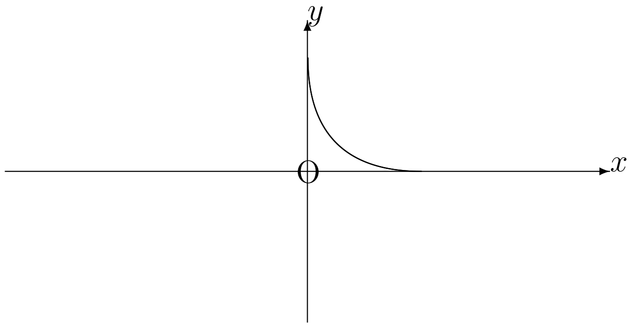
But $x^2 + y^2 = a^2$, and so the first moment becomes

$$\int_0^a 2\pi ax\ \mathrm{d}x = [\pi ax^2]_0^a = \pi a^3.$$

2. Determine the first moments about planes through the origin, (a) perpendicular to the x-axis and (b) perpendicular to the y-axis, of the first quadrant arc of the curve with parametric equations

$$x = a\mathrm{cos}^3\theta, \quad y = a\mathrm{sin}^3\theta.$$

Solution



Firstly, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

Hence, the first moment about the x-axis is given by

$$-\int_{\frac{\pi}{2}}^{0} 2\pi xy\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta,$$

which, on using $\cos^2\theta + \sin^2\theta \equiv 1$, becomes

$$\int_0^{\frac{\pi}{2}} 2\pi a^2\cos^3\theta\sin^3\theta.3a\cos\theta\sin\theta\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} 6\pi a^3\cos^4\theta\sin^4\theta\ \mathrm{d}\theta.$$

Using $2\sin\theta\cos\theta \equiv \sin 2\theta$, the integral reduces to

$$\frac{3\pi a^3}{8}\int_0^{\frac{\pi}{2}} \sin^4 2\theta\ \mathrm{d}\theta,$$

which, by the methods of Unit 12.7, gives

$$\frac{3\pi a^3}{32}\int_0^{\frac{\pi}{2}} \left(1 - 2\cos 4\theta + \frac{1+\cos 8\theta}{2}\right)\ \mathrm{d}\theta = \frac{3\pi a^3}{32}\left[\frac{3\theta}{2} - \frac{\sin 4\theta}{2} + \frac{\sin 8\theta}{16}\right]_0^{\frac{\pi}{2}} = \frac{9\pi a^3}{128}.$$

By symmetry, or by direct integration, the first moment about a plane through the origin, perpendicular to the y-axis is also $\frac{9\pi a^3}{128}$.

13.9.3 THE CENTROID OF A SURFACE OF REVOLUTION

Having calculated the first moment of a surface of revolution about a plane through the origin, perpendicular to the x-axis, it is possible to determine a point, $(\overline{x}, 0)$, on the x-axis with the property that the first moment is given by $S\overline{x}$, where S is the total surface area.

The point is called the **"centroid"** or the **"geometric centre"** of the surface of revolution and, for the surface of revolution of the arc of the curve whose equation is $y = f(x)$, between $x = a$ and $x = b$, the value of $\overline{x}$ is given by

$$\overline{x} = \frac{\int_a^b 2\pi xy\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x}{\int_a^b 2\pi y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x} = \frac{\int_a^b xy\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x}{\int_a^b y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x}.$$

Note:
The centroid effectively tries to concentrate the whole surface at a single point for the purposes of considering first moments. In practice, it corresponds to the position of the centre of mass of a thin sheet, for example, with uniform density.

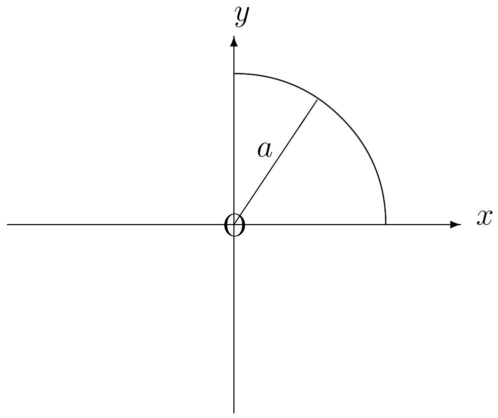
EXAMPLES

1. Determine the position of the centroid of the surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



From Example 1 of Section 13.9.2, we know that the first moment of the surface about a plane through the origin, perpendicular to the the x-axis is equal to πa^3.

Also, the total surface area is

$$\int_0^a 2\pi y\sqrt{1+\frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a 2\pi a\ \mathrm{d}x = 2\pi a^2,$$

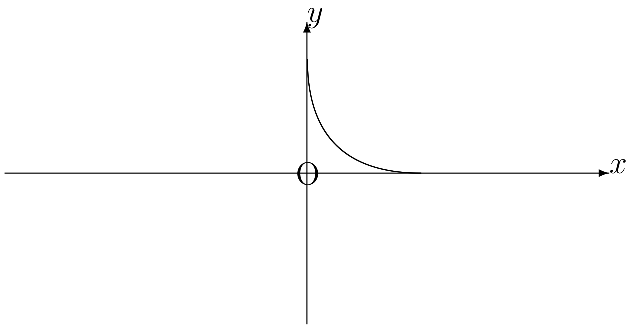
which implies that

$$\overline{x} = \frac{\pi a^3}{2\pi a^2} = \frac{a}{2}.$$

2. Determine the position of the centroid of the surface of revolution (about the x-axis) of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



We know from Example 2 of Section 13.9.2 that that the first moment of the surface about a plane through the origin, perpendicular to the x-axis is equal to $\frac{9\pi a^3}{128}$.
Also, the total surface area is given by

$$-\int_{\frac{\pi}{2}}^{0} 2\pi a\sin^3\theta.3a\cos\theta\sin\theta \ \mathrm{d}\theta = \int_{0}^{\frac{\pi}{2}} 3a^2\sin^4\theta\cos\theta \ \mathrm{d}\theta = 3\pi a^2\left[\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}} = \frac{3\pi a^2}{5}.$$

Thus,

$$\overline{x} = \frac{15a}{128}.$$

13.9.4 EXERCISES

1. Determine the first moment, about a plane through the origin, perpendicular to the x-axis, of the surface of revolution (about the x-axis) of the straight-line segment joining the origin to the point $(3, 4)$.

2. Determine the first moment about a plane through the origin, perpendicular to the x-axis, of the surface of revolution (about the x-axis) of the arc of the curve whose equation is

$$y^2 = 4x,$$

lying between $x = 0$ and $x = 1$.

3. Determine the first moment about a plane through the origin, perpendicular to the y-axis, of the surface of revolution (about the y-axis) of the arc of the curve whose equation is

$$y^2 = 4(x - 1),$$

lying between $y = 2$ and $y = 4$.

4. Determine the first moment, about a plane through the origin, perpendicular to the y-axis, of the surface of revolution (about the y-axis) of the arc of the curve whose parametricc equations are

$$x = 2\cos t, \quad y = 3\sin t,$$

joining the point $(2, 0)$ to the point $(0, 3)$.

5. Determine the position of the centroid of a hollow right-circular cone with height h.

6. For the curve whose equation is

$$9y^2 = x(3 - x)^2,$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 - x}{2\sqrt{x}}.$$

Hence, show that the centroid of the surface obtained when the first quadrant arch of this curve is rotated through 2π radians about the x-axis lies at the point $\left(\frac{5}{4}, 0\right)$.

13.9.5 ANSWERS TO EXERCISES

1.

$$40\pi.$$

2.

$$4\pi\left[\frac{12\sqrt{2}}{5}-\frac{4}{15}\right]\simeq 39.3$$

3.

$$\left[\frac{8\pi}{5}\left(1+\frac{y^2}{4}\right)^{\frac{5}{2}}\right]_2^4 \simeq 41.98$$

4.

$$\left[-\frac{4\pi}{5}\left(4+5\cos^2 t\right)^{\frac{3}{2}}\right]_0^{\frac{\pi}{2}} \simeq 47.75$$

5. Along the central axis, at a distance of $\frac{2h}{3}$ from the vertex.

6.

$$\text{First moment} = \frac{15\pi}{4} \quad \text{Surface Area} = 3\pi.$$