“JUST THE MATHS”

UNIT NUMBER

13.7

INTEGRATION APPLICATIONS 7
(First moments of an area)

by

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UNIT 13.7 - INTEGRATION APPLICATIONS 7

FIRST MOMENTS OF AN AREA

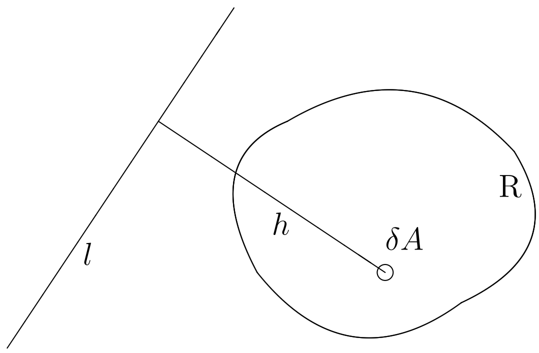
13.7.1 INTRODUCTION

Suppose that R denotes a region (with area A) of the xy-plane of cartesian co-ordinates, and suppose that δA is the area of a small element of this region.

Then the **"first moment"** of R about a fixed line, l, in the plane of R is given by

$$\lim_{\delta A \to 0} \sum_{\mathrm{R}} h\delta A,$$

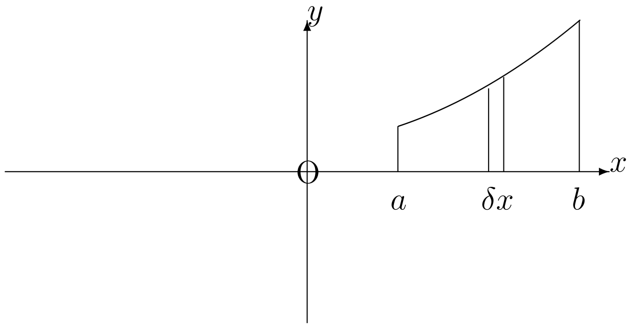
where h is the perpendicular distance, from l, of the element with area, δA.



13.7.2 FIRST MOMENT OF AN AREA ABOUT THE Y-AXIS

Let us consider a region in the first quadrant of the xy-plane, bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The region may divided up into small elements by using a network, consisting of neighbouring lines parallel to the y-axis and neighbouring lines parallel to the x-axis.

But all of the elements in a narrow 'strip' of width δx and height y (parallel to the y-axis) have the same perpendicular distance, x, from the y-axis.

Hence the first moment of this strip about the y-axis is x times the area of the strip; that is, $x(y\delta x)$, implying that the total first moment of the region about the y-axis is given by

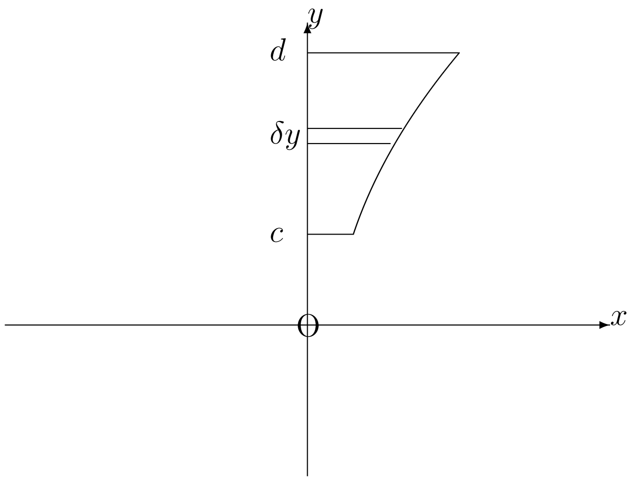
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} xy\delta x = \int_a^b xy \, \mathrm{d}x.$$

Note:
First moments about the x-axis will be discussed mainly in the next section of this Unit; but we note that, for a region of the first quadrant, bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the first moment about the x-axis is given by

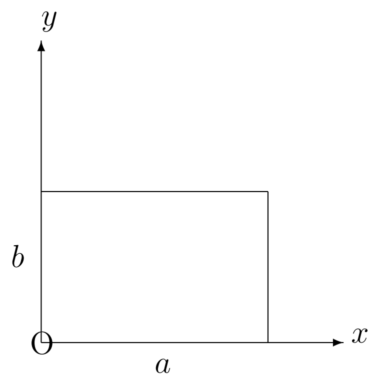
$$\int_c^d yx \, \mathrm{d}y.$$



EXAMPLES

1. Determine the first moment of a rectangular region, with sides of lengths a and b about the side of length b.

   Solution



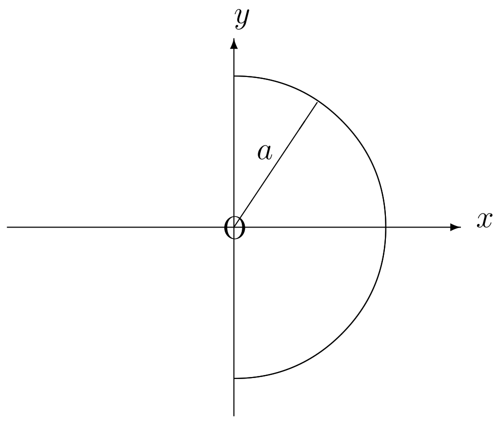
The first moment about the y-axis is given by

$$\int_0^a xb \, \mathrm{d}x = \left[\frac{x^2 b}{2}\right]_0^a = \frac{1}{2}a^2 b.$$

2. Determine the first moment about the y-axis of the semi-circular region, bounded in the first and fourth quadrants by the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$

Solution



Since there will be equal contributions from the upper and lower halves of the region, the first moment about the y-axis is given by

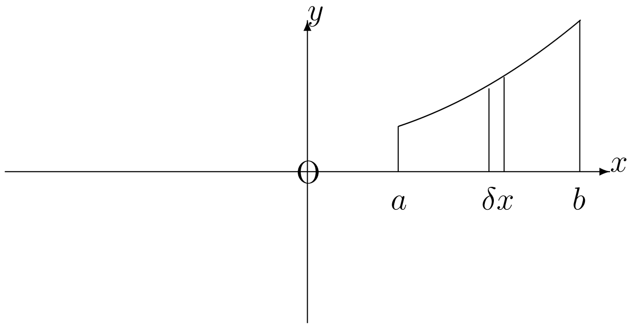
$$2\int_0^a x\sqrt{a^2 - x^2}\ \mathrm{d}x = \left[-\frac{2}{3}(a^2 - x^2)^{\frac{3}{2}}\right]_0^a = \frac{2}{3}a^3.$$

Note:
Although first moments about the x-axis will be discussed mainly in the next section of this Unit, we note that the symmetry of the above region shows that its first moment about the x-axis would be zero; this is because, for each $y(x\delta y)$, there will be a corresponding $-y(x\delta y)$ in calculating the first moments of the strips parallel to the x-axis.

13.7.3 FIRST MOMENT OF AN AREA ABOUT THE X-AXIS

In the first example of the previous section, a formula was established for the first moment of a rectangular region about one of its sides. This result may now be used to determine the first moment about the x-axis of a region enclosed in the first quadrant by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



If a narrow strip, of width δx and height y, is regarded as approximately a rectangle, its first moment about the x-axis is $\frac{1}{2}y^2\delta x$. Hence, the first moment of the whole region about the x-axis is given by

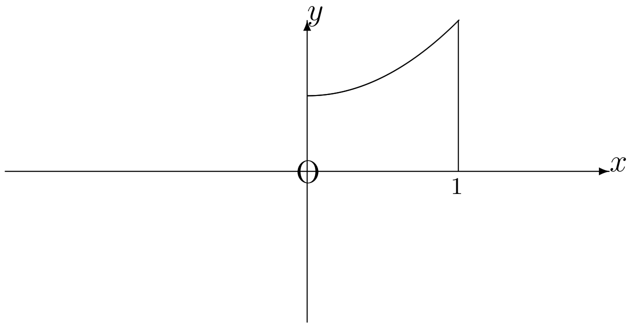
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \frac{1}{2}y^2\delta x = \int_a^b \frac{1}{2}y^2 \, \mathrm{d}x.$$

EXAMPLES

1. Determine the first moment about the x-axis of the region, bounded in the first quadrant, by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = x^2 + 1.$$

Solution

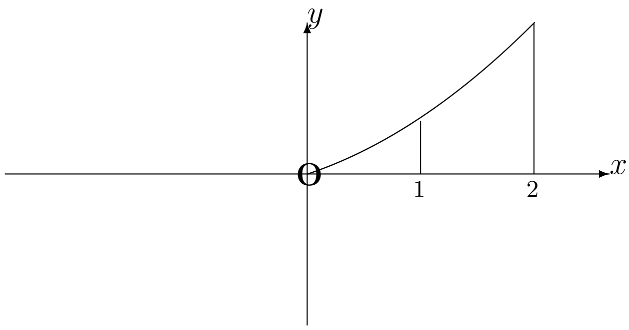


$$\text{First moment } = \int_0^1 \frac{1}{2}(x^2+1)^2 \, \mathrm{d}x = \frac{1}{2}\int_0^1 (x^4+2x^2+1) \, \mathrm{d}x = \frac{1}{2}\left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_0^1 = \frac{28}{15}.$$

2. Determine the first moment about the x-axis of the region, bounded in the first quadrant, by the x-axis, the lines $x = 1$, $x = 2$ and the curve

$$y = xe^x.$$

Solution



$$\text{First moment } = \int_1^2 \frac{1}{2}x^2e^{2x} \, \mathrm{d}x$$

$$= \frac{1}{2}\left(\left[x^2\frac{e^{2x}}{2}\right]_1^2 - \int_1^2 xe^{2x}\ \mathrm{d}x\right)$$

$$= \frac{1}{2}\left(\left[x^2\frac{e^{2x}}{2}\right]_1^2 - \left[x\frac{e^{2x}}{2}\right]_1^2 + \int_1^2 \frac{e^{2x}}{2}\ \mathrm{d}x\right).$$

That is,

$$\frac{1}{2}\left[x^2\frac{e^{2x}}{2} - x\frac{e^{2x}}{2} + \frac{e^{2x}}{4}\right]_1^2 = \frac{5e^4 - e^2}{8} \simeq 33.20$$

13.7.4 THE CENTROID OF AN AREA

Having calculated the first moments of a two dimensional region about both the x-axis and the y-axis, it is possible to determine a point, $(\overline{x}, \overline{y})$, in the xy-plane with the property that

(a) The first moment about the y-axis is given by $A\overline{x}$, where A is the total area of the region;

and

(b) The first moment about the x-axis is given by $A\overline{y}$, where A is the toal area of the region.

The point is called the **"centroid"** or the **"geometric centre"** of the region and, in the case of a region bounded, in the first quadrant, by the x-axis, the lines $x = a$, $x = b$ and the curve $y = f(x)$, its co-ordinates are given by

$$\overline{x} = \frac{\int_a^b\ xy\ \mathrm{d}x}{\int_a^b\ y\mathrm{d}x} \quad \text{and} \quad \overline{y} = \frac{\int_a^b\ \frac{1}{2}y^2\ \mathrm{d}x}{\int_a^b\ y\ \mathrm{d}x}.$$

Notes:

(i) The first moment of an area, about an axis through its centroid will, by definition, be zero. In particular, if we take the y-axis to be parallel to the given axis, with x as the perpendicular distance from an element, δA, to the y-axis, the first moment about the given axis will be

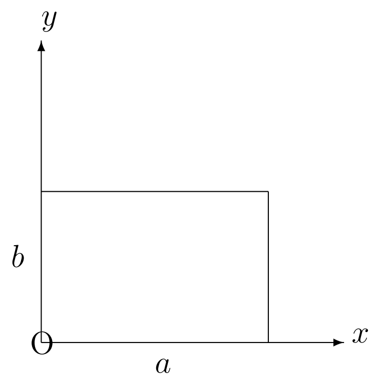
$$\sum_{\mathrm{R}} (x - \overline{x})\delta A = \sum_{\mathrm{R}} x\delta A - \overline{x}\sum_{\mathrm{R}} \delta A = A\overline{x} - A\overline{x} = 0.$$

(ii) The centroid effectively tries to concentrate the whole area at a single point for the purposes of considering first moments. In practice, it corresponds to the position of the centre of mass for a thin plate with uniform density, whose shape is that of the region which we have been considering.

EXAMPLES

1. Determine the position of the centroid of a rectangular region with sides of lengths, a and b.

   Solution

   The area of the rectangle is ab and, from Example 1 in section 13.7.2, the first moments about the y-axis and the x-axis are $\frac{1}{2}a^2b$ and $\frac{1}{2}b^2a$, respectively.

   Hence,

   $$\overline{x} = \frac{\frac{1}{2}a^2b}{ab} = \frac{1}{2}a$$

   and

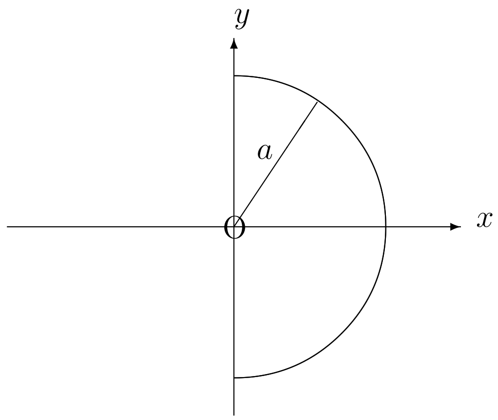
   $$\overline{y} = \frac{\frac{1}{2}b^2a}{ab} = \frac{1}{2}b,$$

   as we would expect for a rectangle.

2. Determine the position of the centroid of the semi-circular region bounded, in the first and fourth quadrants, by the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$

Solution



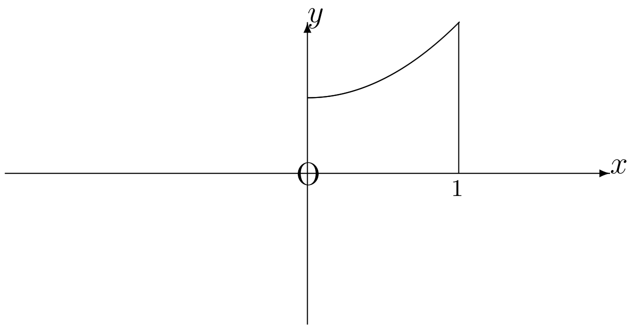
The area of the semi-circular region is $\frac{1}{2}\pi a^2$ and so, from Example 2, in section 13.7.2,

$$\overline{x} = \frac{\frac{2}{3}a^3}{\frac{1}{2}\pi a^2} = \frac{4a}{3\pi} \text{ and } \overline{y} = 0.$$

3. Determine the position of the centroid of the region, bounded in the first quadrant, by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = x^2 + 1.$$

Solution



The first moment about the y-axis is given by

$$\int_0^1 x(x^2+1)\ \mathrm{d}x = \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^1 = \frac{3}{4}.$$

The area is given by

$$\int_0^1 (x^2+1)\ \mathrm{d}x = \left[\frac{x^3}{3} + x\right]_0^1 = \frac{4}{3}.$$

Hence,

$$\overline{x} = \frac{3}{4} \div \frac{4}{3} = 1.$$

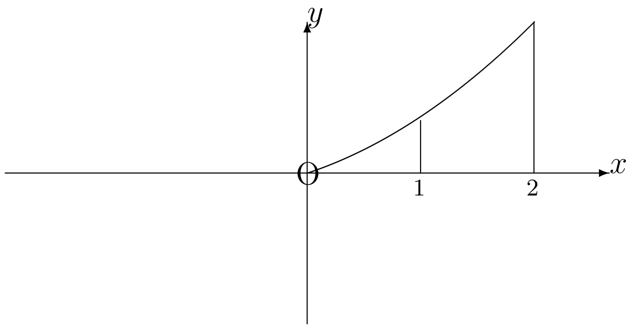
The first moment about the x-axis is $\frac{28}{15}$, from Example 1 in section 13.7.3; and, therefore,

$$\overline{y} = \frac{28}{15} \div \frac{4}{3} = \frac{7}{5}.$$

4. Determine the position of the centroid of the region bounded in the first quadrant by the x-axis, the lines $x = 1$, $x = 2$ and the curve whose equation is

$$y = xe^x.$$

Solution



The first moment about the y-axis is given by

$$\int_1^2 x^2e^x \, \mathrm{d}x = \left[x^2e^x - 2xe^x + 2e^x\right]_1^2 \simeq 12.06,$$

using integration by parts (twice).
The area is given by

$$\int_1^2 xe^x \, \mathrm{d}x = [xe^x - e^x]_1^2 \simeq 7.39$$

using integration by parts (once).
Hence,

$$\overline{x} \simeq 12.06 \div 7.39 \simeq 1.63$$

The first moment about the x-axis is approximately 33.20, from Example 2 in section 13.7.3; and so,

$$\overline{y} \simeq 33.20 \div 7.39 \simeq 4.47$$

13.7.5 EXERCISES

Determine the position of the centroid of each of the following regions of the xy-plane:

1. Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is

$$x^2 + y^2 = a^2.$$

2. Bounded by the line $x = 1$ and the semi-circle whose equation is

$$(x-1)^2 + y^2 = 4, \ x > 1.$$

3. Bounded in the fourth quadrant by the x-axis, the y-axis and the curve whose equation is

$$y = 2x^2 - 1.$$

4. Bounded in the first quadrant by the x-axis and the curve whose equation is

$$y = \sin x.$$

5. Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = xe^{-2x}.$$

13.7.6 ANSWERS TO EXERCISES

1.

$$\left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right).$$

2.

$$\left(\frac{11}{3\pi}, 0\right).$$

3.

$$\left(\frac{3\sqrt{2}}{16}, -\frac{13}{20}\right).$$

4.

$$\left(\frac{\pi}{2}, \frac{\pi}{8}\right).$$

5.

$$(0.28, 0.04).$$