"JUST THE MATHS"

UNIT NUMBER

13.6

INTEGRATION APPLICATIONS 6
(First moments of an arc)

by

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UNIT 13.6 - INTEGRATION APPLICATIONS 6

FIRST MOMENTS OF AN ARC

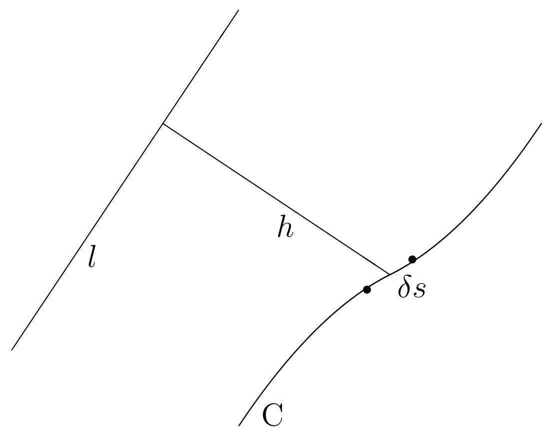
13.6.1 INTRODUCTION

Suppose that C denotes an arc (with length s) in the xy-plane of cartesian co-ordinates; and suppose that δs is the length of a small element of this arc.

Then the **"first moment"** of C about a fixed line, l, in the plane of C is given by

$$\lim_{\delta s \to 0} \sum_{\text{C}} h\delta s,$$

where h is the perpendicular distance, from l, of the element with length δs.

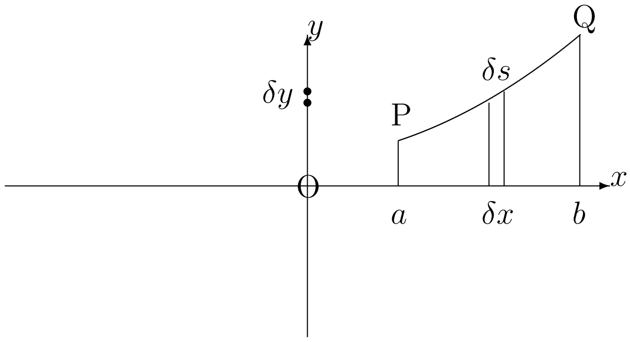


13.6.2 FIRST MOMENT OF AN ARC ABOUT THE Y-AXIS

Let us consider an arc of the curve, whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

The first moment of each element about the y-axis is x times the length of the element; that is $x\delta s$, implying that the total first moment of the arc about the y-axis is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} x\delta s.$$

But, from Pythagoras' Theorem,

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x,$$

so that the first moment of the arc becomes

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} x\sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x = \int_a^b x\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, we may conclude that the first moment of the arc about the y-axis is given by

$$\pm \int_{t_1}^{t_2} x \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \, \mathrm{d}t,$$

according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

13.6.3 FIRST MOMENT OF AN ARC ABOUT THE X-AXIS

(a) For an arc whose equation is

$$y = f(x),$$

contained between $x = a$ and $x = b$, the first moment about the x-axis will be

$$\int_a^b y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \, \mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, the first moment of the arc about the x-axis is given by

$$\pm \int_{t_1}^{t_2} y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \, \mathrm{d}t,$$

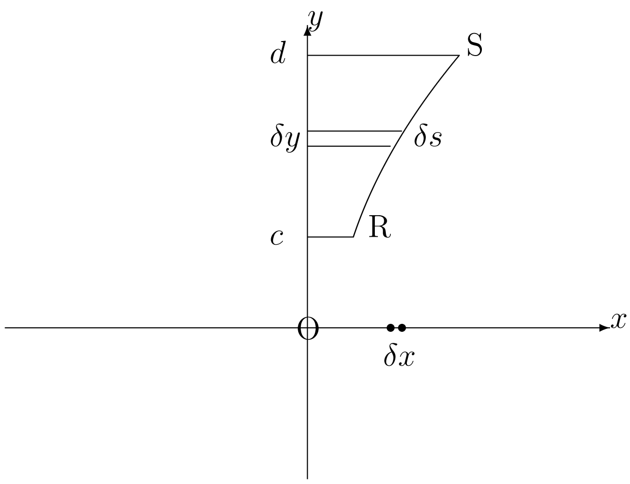
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in section 13.6.2 so that the first moment of the arc about the x-axis is given by

$$\int_c^d y\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$



Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, we may conclude that the first moment of the arc about the x-axis is given by

$$\pm\int_{t_1}^{t_2} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative and where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$.

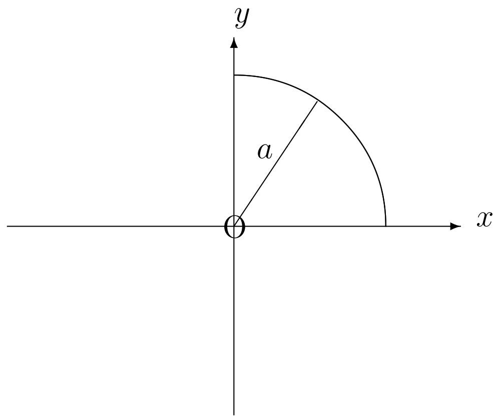
EXAMPLES

1. Determine the first moments about the x-axis and the y-axis of the arc of the circle, with equation

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



Using implicit differentiation, we have

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

and hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

The first moment of the arc about the y-axis is therefore given by

$$\int_0^a x\sqrt{1 + \frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a \frac{x}{y}\sqrt{x^2 + y^2}\ \mathrm{d}x.$$

But $x^2 + y^2 = a^2$ and $y = \sqrt{a^2 - x^2}$.

Hence,

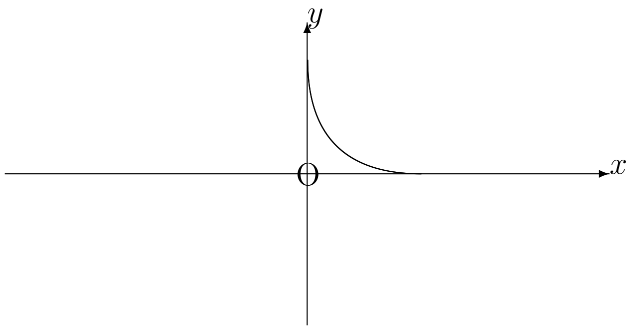
$$\text{first moment} = \int_0^a \frac{ax}{\sqrt{a^2 - x^2}}\ \mathrm{d}x = \left[-a\sqrt{(a^2 - x^2)}\right]_0^a = a^2.$$

By symmetry, the first moment of the arc about the x-axis will also be a^2.

2. Determine the first moments about the x-axis and the y-axis of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



Firstly, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \text{ and } \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

Hence, the first moment about the x-axis is given by

$$-\int_{\frac{\pi}{2}}^{0} y\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta,$$

which, on using $\cos^2\theta + \sin^2\theta \equiv 1$, becomes

$$\int_0^{\frac{\pi}{2}} a\sin^3\theta.3a\cos\theta\sin\theta\ \mathrm{d}\theta$$

$$= 3a^2\int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\ \mathrm{d}\theta$$

$$= 3a^2\left[\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}} = \frac{3a^2}{5}.$$

Similarly, the first moment of the arc about the y-axis is given by

$$\int_0^{\frac{\pi}{2}} x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} a\cos^3\theta.(3a\cos\theta\sin\theta)\ \mathrm{d}\theta$$

$$= 3a^2\int_0^{\frac{\pi}{2}} \cos^4\theta\sin\theta\ \mathrm{d}\theta = 3a^2\left[-\frac{\cos^5\theta}{5}\right]_0^{\frac{\pi}{2}} = \frac{3a^2}{5},$$

though, again, this second result could be deduced, by symmetry, from the first.

13.6.4 THE CENTROID OF AN ARC

Having calculated the first moments of an arc about both the x-axis and the y-axis it is possible to determine a point, $(\overline{x}, \overline{y})$, in the xy-plane with the property that

(a) The first moment about the y-axis is given by $s\overline{x}$, where s is the total length of the arc;

and

(b) The first moment about the x-axis is given by $s\overline{y}$, where s is the total length of the arc.

The point is called the **"centroid"** or the **"geometric centre"** of the arc and, for an arc of the curve with equation $y = f(x)$, between $x = a$ and $x = b$, its co-ordinates are given by

$$\overline{x} = \frac{\int_a^b x\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x}{\int_a^b \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x} \quad \text{and} \quad \overline{y} = \frac{\int_a^b y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x}{\int_a^b \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x}.$$

Notes:

(i) The first moment of an arc about an axis through its centroid will, by definition, be zero. In particular, if we take the y-axis to be parallel to the given axis, with x as the perpendicular distance from an element, δs, to the y-axis, the first moment about the given axis will be

$$\sum_{\mathrm{C}} (x-\overline{x})\delta s = \sum_{\mathrm{C}} x\delta s - \overline{x}\sum_{\mathrm{C}} \delta s = s\overline{x} - s\overline{x} = 0.$$

(ii) The centroid effectively tries to concentrate the whole arc at a single point for the purposes of considering first moments. In practice, it corresponds, for example, to the position of the centre of mass of a thin wire with uniform density.

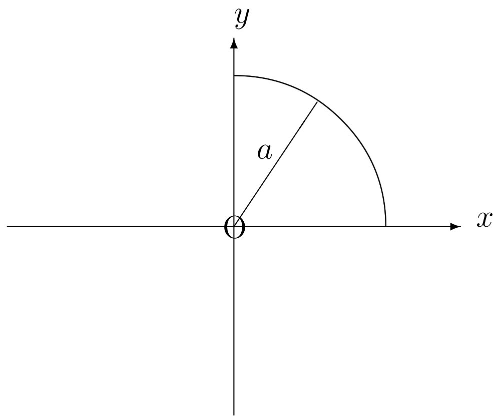
EXAMPLES

1. Determine the cartesian co-ordinates of the centroid of the arc of the circle, with equation

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



From an earlier example in this unit, we know that the first moments of the arc about the x-axis and the y-axis are both equal to a^2.

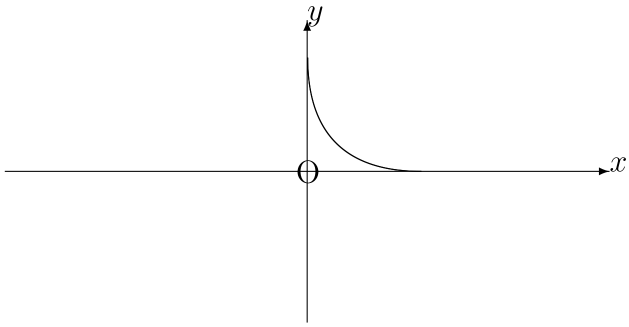
Also, the length of the arc is $\frac{\pi a}{2}$, which implies that

$$\overline{x} = \frac{2a}{\pi} \quad \text{and} \quad \overline{y} = \frac{2a}{\pi}.$$

2. Determine the cartesian co-ordinates of the centroid of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



From an earlier example in this unit, we know that

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \ \text{ and } \ \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta$$

and that the first moments of the arc about the x-axis and the y-axis are both equal to $\frac{3a^2}{5}$.
Also, the length of the arc is given by

$$-\int_{\frac{\pi}{2}}^{a} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta.$$

This simplifies to

$$3a\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\ \mathrm{d}\theta = 3a\left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{3a}{2}.$$

Thus,

$$\overline{x} = \frac{2a}{5} \ \text{ and } \ \overline{y} = \frac{2a}{5}.$$

13.6.5 EXERCISES

1. Determine the first moment about the y-axis of the arc of the curve with equation

$$y = x^2,$$

lying between $x = 0$ and $x = 1$.

2. Determine the first moment about the x-axis of the arc of the curve with equation

$$x = 5y^2,$$

lying between $y = 0.1$ and $y = 0.5$.

3. Determine the first moment about the x-axis of the arc of the curve with equation

$$y = 2\sqrt{x},$$

lying between $x = 3$ and $x = 24$.

4. Verify, using integration, that the centroid of the straight line segment, defined by the equation

$$y = 3x + 2,$$

from $x = 0$ to $x = 1$, lies at its centre point.

5. Determine the cartesian co-ordinates of the centroid of the arc of the circle given parametrically by

$$x = 5\cos\theta, \quad y = 5\sin\theta,$$

from $\theta = -\frac{\pi}{6}$ to $\theta = \frac{\pi}{6}$.

6. For the curve whose equation is

$$9y^2 = x(3 - x)^2,$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 - x}{2\sqrt{x}}.$$

Hence show that the centroid of the first quadrant arch of this curve lies at the point $\left(\frac{7}{5}, \frac{\sqrt{3}}{4}\right)$.

13.6.6 ANSWERS TO EXERCISES

1.

$$\frac{5\sqrt{5}-1}{12} \simeq 0.85$$

2.

$$\frac{13\sqrt{26}-\sqrt{2}}{150} \simeq 0.43$$

3.

$$156.$$

4.

$$\overline{x} = \frac{1}{2} \quad \text{and} \quad \overline{y} = \frac{7}{2}.$$

5.

$$\overline{x} = \frac{15}{\pi} \simeq 4.77, \quad \overline{y} = 0.$$