"JUST THE MATHS"

UNIT NUMBER

13.5

INTEGRATION APPLICATIONS 5
(Surfaces of revolution)

by

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UNIT 13.5 - INTEGRATION APPLICATIONS 5

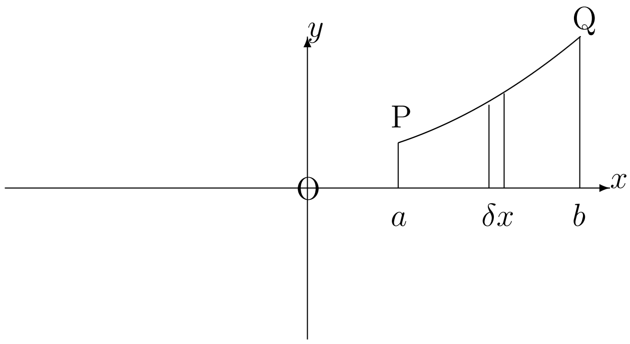
SURFACES OF REVOLUTION

13.5.1 SURFACES OF REVOLUTION ABOUT THE X-AXIS

The problem, in this unit, is to calculate the surface area obtained when the arc of the curve, with equation

$$y = f(x),$$

joining the two points, P and Q, on the curve, at which $x = a$ and $x = b$, is rotated through 2π radians about the x-axis or the y-axis.



For two neighbouring points along the arc, the part of the curve joining them may be considered, approximately, as a straight line segment.

Hence, if these neighbouring points are separated by distances of δx and δy, parallel to the x-axis and the y-axis, respectively, then the length, δs, of arc between them is given, approximately, by

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x,$$

using Pythagoras's Theorem.

When the arc, of length δs, is rotated through 2π radians about the x-axis, it generates a thin band whose area is, approximately,

$$2\pi y \delta s = 2\pi y \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x.$$

The total surface area, S, is thus given by

$$S = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi y \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x.$$

That is,

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \, \mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered. If not, then the above line needs to be prefixed by a negative sign.

From the technique of integration by substitution,

$$\int_a^b 2\pi y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x = \int_{t_1}^{t_2} 2\pi y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

We may conclude that

$$S = \pm\int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

acccording as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

EXAMPLES

1. A curve has equation

   $$y^2 = 2x.$$

   Determine the surface area obtained when the arc of the curve between the point $(2, 2))$ and the point $(8, 4)$ is rotated through 2π radians about the x-axis.

   Solution

   We may write the equation of the arc of the curve in the form

   $$y = \sqrt{2x} = \sqrt{2}x^{\frac{1}{2}};$$

   and so,

   $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\sqrt{2}x^{-\frac{1}{2}} = \frac{1}{\sqrt{2x}}.$$

   Hence,

   $$S = \int_2^8 2\pi\sqrt{2x}\sqrt{1+\frac{1}{2x}}\ \mathrm{d}x = \int_2^8 \sqrt{2x+1}\ \mathrm{d}x = \left[\frac{(2x+1)^{\frac{3}{2}}}{3}\right]_2^8.$$

   Thus,

   $$S = \frac{17^{\frac{3}{2}}}{3} - \frac{5^{\frac{3}{2}}}{3} \simeq 19.64$$

2. A curve is given parametrically by

$$x = \sqrt{2}\cos\theta, \quad y = \sqrt{2}\sin\theta.$$

Determine the surface area obtained when the arc of the curve between the point $(0, \sqrt{2})$ and the point $(1, 1)$ is rotated through 2π radians about the x-axis.

Solution

The parameters of the two points are $\frac{\pi}{2}$ and $\frac{\pi}{4}$, respectively; and, since

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\sqrt{2}\sin\theta \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = \sqrt{2}\cos\theta,$$

we have

$$S = -\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 2\sqrt{2}\pi\sin\theta\sqrt{2\sin^2\theta + 2\cos^2\theta}\ \mathrm{d}\theta = -\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 4\pi\sin\theta\ \mathrm{d}\theta.$$

Thus,

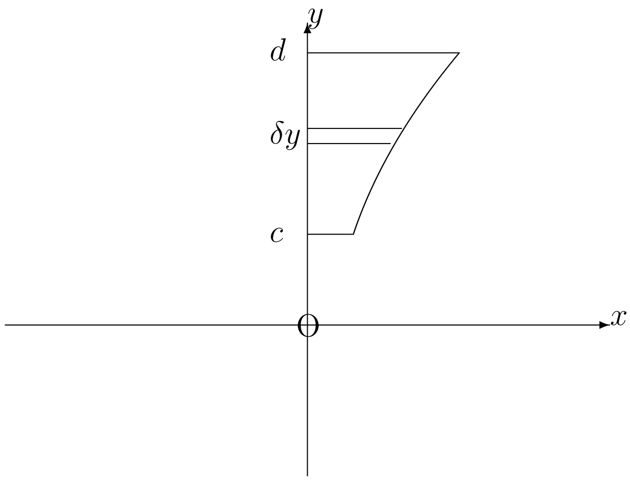
$$S = -\left[-4\pi\cos\theta\right]_{\frac{\pi}{2}}^{\frac{\pi}{4}} = \frac{4\pi}{\sqrt{2}} \simeq 8.89$$

13.5.2 SURFACES OF REVOLUTION ABOUT THE Y-AXIS

For a curve whose equation is of the form $x = g(y)$, the surface of revolution about the y-axis of an arc joining the two points at which $y = c$ and $y = d$ is given by

$$S = \int_a^b 2\pi x\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$

We simply reverse the roles of x and y in the previous section.



Alternatively, if the curve is given parametrically,

$$S = \pm \int_{t_1}^{t_2} 2\pi x \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

acccording as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative.

EXAMPLE

If the arc of the parabola, with equation

$$x^2 = 2y,$$

joining the two points $(2, 2)$ and $(4, 8)$, is rotated through 2π radians about the y-axis, determine the surface area obtained.

Solution

Using the result from the previous section, the surface area obtained is given by

$$S = \int_2^8 2\pi\sqrt{2y}\sqrt{1 + \frac{1}{2y}}\ \mathrm{d}y \simeq 19.64$$

13.5.3 EXERCISES

1. Use a straight line through the origin to determine the surface area of a right-circular cone with height, h, and base radius, r.

2. Determine the surface area obtained when the arc of the curve $x = y^3$, between $y = 0$ and $y = 1$, is rotated through 2π radians about the y-axis.

3. A curve is given parametrically by

$$x = t - \sin t, \quad y = 1 - \cos t.$$

   Determine the surface area obtained when the arc of the curve between the point where $t = 0$ and the point where $t = \frac{\pi}{2}$ is rotated through 2π radians about the x-axis.

   State your answer correct to three places of decimals.

4. A curve is given parametrically by

$$x = 4(\cos\theta + \theta\sin\theta), \quad y = 4(\sin\theta - \theta\cos\theta).$$

   Determine the surface area obtained when the arc of the curve between the point where $\theta = 0$ and the point where $\theta = \frac{\pi}{2}$ is rotated through 2π radians about the x-axis.

5. A curve is given parametrically by

$$x = e^u\cos u, \quad y = e^u\sin u.$$

   Determine the surface area obtained when the arc of the curve between the point where $u = 0$ and the point where $u = \frac{\pi}{4}$ is rotated through 2π radians about the y-axis.

   State your answer correct to three places of decimals.

13.5.4 ANSWERS TO EXERCISES

1.

$$\pi r\sqrt{r^2 + h^2}.$$

2.

$$\frac{\pi(10\sqrt{10} - 1)}{27} \simeq 3.56$$

3.

$$3.891$$

4.

$$32\pi\left(3 - \left(\frac{\pi}{2}\right)^2\right) \simeq 53.54$$

5.

$$1.037$$