"JUST THE MATHS"

UNIT NUMBER

13.4

INTEGRATION APPLICATIONS 4
(Lengths of curves)

by

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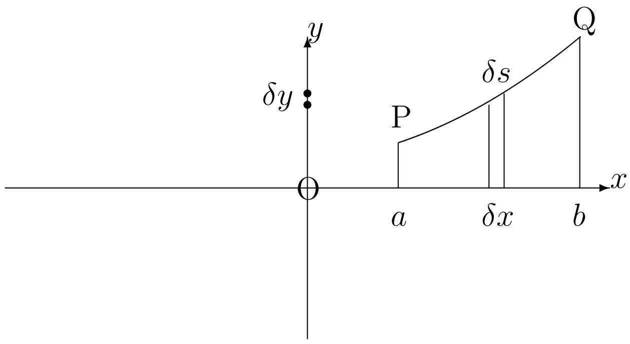
UNIT 13.4 - INTEGRATION APPLICATIONS 4 - LENGTHS OF CURVES

13.4.1 THE STANDARD FORMULAE

The problem, in this unit, is to calculate the length of the arc of the curve with equation

$$y = f(x),$$

joining the two points, P and Q, on the curve, at which $x = a$ and $x = b$.



For two neighbouring points along the arc, the part of the curve joining them may be considered, approximately, as a straight line segment.

Hence, if these neighbouring points are separated by distances of δx and δy, parallel to the x-axis and the y-axis respectively, then the length, δs, of arc between them is given, approximately, by

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x,$$

using Pythagoras's Theorem.

The total length, s, of arc is thus given by

$$s = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x.$$

That is,

$$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x.$$

Notes:
(i) If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered. If not, then the above line needs to be prefixed by a negative sign.

From the technique of integration by substitution,

$$\int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x = \int_{t_1}^{t_2} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

We may conclude that

$$s = \pm \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

acccording as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(ii) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y, so that the length of the arc is given by

$$s = \int_c^d \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$

EXAMPLES

1. A curve has equation

$$9y^2 = 16x^3.$$

Determine the length of the arc of the curve between the point $\left(1, \frac{4}{3}\right)$ and the point $\left(4, \frac{32}{3}\right)$.

Solution

We may write the equation of the curve in the form

$$y = \frac{4x^{\frac{3}{2}}}{3};$$

and so,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x^{\frac{1}{2}}.$$

Hence,

$$s = \int_1^4 \sqrt{1+4x}\ \mathrm{d}x = \left[\frac{(1+4x)^{\frac{3}{2}}}{6}\right]_1^4 = \frac{17^{\frac{3}{2}}}{6} - \frac{5^{\frac{3}{2}}}{6} \simeq 13.55$$

2. A curve is given parametrically by

$$x = t^2 - 1, \quad y = t^3 + 1.$$

Determine the length of the arc of the curve between the point where $t = 0$ and the point where $t = 1$.

Solution

Since

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2,$$

we have

$$s = \int_0^1 \sqrt{4t^2 + 6t^4}\ \mathrm{d}t = \int_0^1 t\sqrt{4 + 6t^2}\ \mathrm{d}t = \left[\frac{1}{18}\left(4 + 6t^2\right)^{\frac{3}{2}}\right]_0^1 = \frac{1}{18}\left(10^{\frac{3}{2}} - 8\right) \simeq 1.31$$

13.4.2 EXERCISES

1. A straight line has equation

$$y = 3x + 2.$$

Use (a) elementary trigonometry and (b) definite integration to determine the length of the line segment joining the point where $x = 3$ and the point where $x = 7$.

2. A curve has equation

$$y = \frac{1}{2}x^2 - \frac{1}{4}\ln x.$$

Determine the length of the arc of the curve between $x = 1$ and $x = e$.

3. A curve has equation

$$x = 2(y + 3)^{\frac{3}{2}}.$$

Determine the length of the arc of the curve between $y = -2$ and $y = 1$, stating your answer in decimals correct to four significant figures.

4. A curve is given parametrically by

$$x = t - \sin t, \quad y = 1 - \cos t.$$

Determine the length of the arc of the curve between the point where $t = 0$ and the point where $t = 2\pi$.

5. A curve is given parametrically by

$$x = 4(\cos\theta + \theta\sin\theta), \quad y = 4(\sin\theta - \theta\cos\theta).$$

Determine the length of the arc of the curve between the point where $\theta = 0$ and the point where $\theta = \frac{\pi}{4}$.

6. A curve is given parametrically by

$$x = e^u \sin u, \quad y = e^u \cos u.$$

Determine the length of the arc of the curve between the point where $u = 0$ and the point where $u = 1$.

13.4.3 ANSWERS TO EXERCISES

1.

$$4\sqrt{10} \simeq 12.65$$

2.

$$\frac{2e^2 - 1}{4} \simeq 3.44$$

3.

$$14.33$$

4.

$$8.$$

5.

$$\frac{\pi^2}{8}.$$

6.

$$\sqrt{2}(e - 1) \simeq 2.43$$