"JUST THE MATHS"

UNIT NUMBER

13.3

INTEGRATION APPLICATIONS 3
(Volumes of revolution)

by

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UNIT 13.3 - INTEGRATION APLICATIONS 3

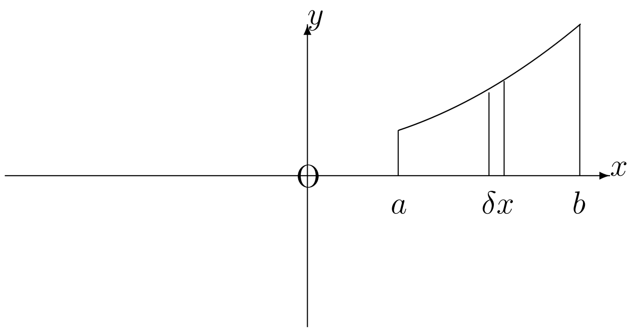
VOLUMES OF REVOLUTION

13.3.1 VOLUMES OF REVOLUTION ABOUT THE X-AXIS

Suppose that the area between a curve whose equation is

$$y = f(x)$$

and the x-axis, from $x = a$ to $x = b$, lies wholly above the x-axis; suppose, also, that this area is rotated through 2π radians about the x-axis. Then a solid figure is obtained whose volume may be determined as an application of definite integration.



When a narrow strip of width, δx, and height, y, is rotated through 2π radians about the x-axis, we obtain a disc whose volume, δV, is given approximately by

$$\delta V \simeq \pi y^2 \delta x.$$

Thus, the total volume, V, obtained is given by

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \delta x.$$

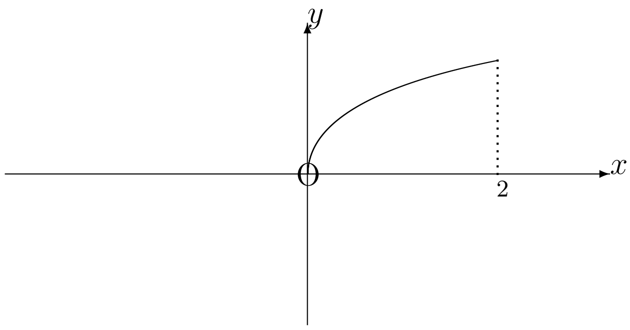
That is,

$$V = \int_a^b \pi y^2 \, \mathrm{d}x.$$

EXAMPLE

Determine the volume obtained when the area, bounded in the first quadrant by the x-axis, the y-axis, the straight line, $x = 2$, and the parabola, $y^2 = 8x$, is rotated through 2π radians about the x-axis.

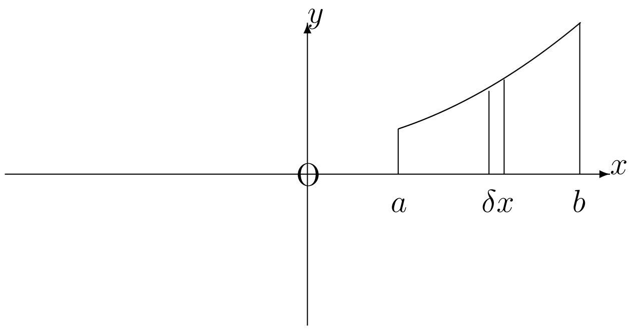
Solution



$$V = \int_0^2 \pi \times 8x \, \mathrm{d}x = [4\pi x^2]_0^2 = 16\pi.$$

13.3.2 VOLUMES OF REVOLUTION ABOUT THE Y-AXIS

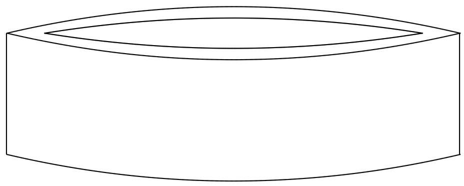
First we consider the same diagram as in the previous section:



This time, if the narrow strip of width, δx, is rotated through 2π radians about the y-axis, we obtain, approximately, a cylindrical shell of internal radius, x, external radius, $x + \delta x$ and height, y.

The volume, δV, of the shell is thus given by

$$\delta V \simeq 2\pi xy\delta x.$$



The total volume is given by

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi xy\delta x.$$

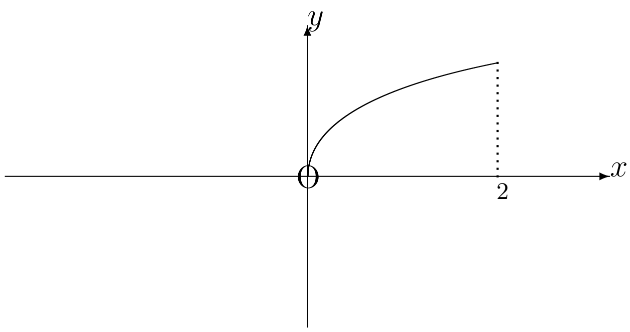
That is,

$$V = \int_a^b 2\pi xy \ \mathrm{d}x.$$

EXAMPLE

Determine the volume obtained when the area, bounded in the first quadrant by the x-axis, the y-axis, the straight line $x = 2$ and the parabola $y^2 = 8x$ is rotated through 2π radians about the y-axis.

Solution



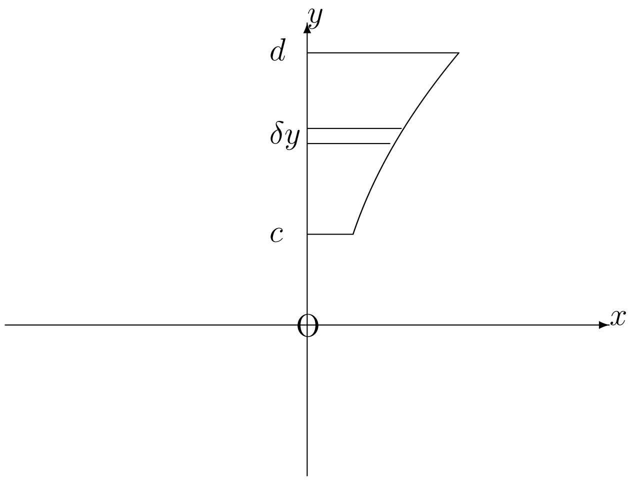
$$V = \int_0^2 2\pi x \times \sqrt{8x}\ \mathrm{d}x.$$

In other words,

$$V = \pi 4\sqrt{2}\int_0^2 x^{\frac{3}{2}}\mathrm{d}x = \pi 4\sqrt{2}\left[\frac{2x^{\frac{5}{2}}}{5}\right]_0^2 = \frac{64\pi}{5}.$$

Note:

It may be required to find the volume of revolution about the y-axis of an area which is contained between a curve and the y-axis from $y = c$ to $y = d$.



But here we simply interchange the roles of x and y in the original formula for rotation about the x-axis; that is

$$V = \int_c^d \pi x^2 \text{ d}y.$$

Similarly, the volume of rotation of the above area about the x-axis is given by

$$V = \int_c^d 2\pi y x \text{ d}y.$$

13.3.3 EXERCISES

1. By using a straight line through the origin, obtain a formula for the volume, V, of a solid right-circular cone with height, h, and base radius, r.
2. Determine the volume obtained when the segment straight line

$$y = 5 - 4x,$$

   lying between $x = 0$ and $x = 1$, is rotated through 2π radians about (a) the x-axis and (b) the y-axis.
3. Determine the volume obtained when the part of the curve

$$y = \cos 3x,$$

   lying between $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$, is rotated through 2π radians about the x-axis.

4. Determine the volume obtained when the part of the curve
$$y = \frac{1}{x\sqrt{2+x}},$$
lying between $x = 2$ and $x = 7$, is rotated through 2π radians about the x-axis.

5. Determine the volume obtained when the part of the curve
$$y = \frac{1}{(x-1)(x-5)},$$
lying between $x = 6$ and $x = 8$, is rotated through 2π radians about the y-axis.

6. Determine the volume obtained when the part of the curve
$$x = ye^{-y},$$
lying between $y = 0$ and $y = 1$, is rotated through 2π radians about the y-axis.

7. Determine the volume obtained when the part of the curve
$$y = \sin 2x,$$
lying between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$, is rotated through 2π radians about the y-axis.

8. Determine the volume obtained when the part of the curve
$$y = x(1-x^3)^{\frac{1}{4}},$$
lying between $x = 0$ and $x = 1$, is rotated through 2π radians about the x-axis.

9. Determine the volume obtained when the part of the curve
$$x = (4-y^2)^2,$$
lying between $y = 1$ and $y = 2$, is rotated through 2π radians about the x-axis.

10. Determine the volume obtained when the part of the curve
$$y = x\sec(x^3),$$
lying between $x = 0$ and $x = 0.5$, is rotated through 2π radians about the x-axis.

11. Determine the volume obtained when the part of the curve
$$y = \frac{1}{x^2-1},$$
lying between $x = 2$ and $x = 3$ is rotated through 2π radians about the y-axis.

13.3.4 ANSWERS TO EXERCISES

1.

$$V = \frac{1}{3}\pi r^2 h.$$

2.

$$\text{(a)} \quad \frac{\pi}{3} \simeq 1.047 \quad \text{(b)} \quad \frac{7\pi}{3} \simeq 7.330$$

3.

$$\frac{\pi^2}{12} \simeq 0.822$$

4.

0.214 approximately.

5.

8.010 approximately.

6.

0.254 approximately.

7.

3.364 approximately.

8.

$$\frac{2\pi}{9} \simeq 0.698$$

9.

$$9\pi \simeq 28.274$$

10.

0.132 approximately.

11.

3.081 approximately.