"JUST THE MATHS"

UNIT NUMBER

13.2

INTEGRATION APPLICATIONS 2
(Mean values)
&
(Root mean square values)

by

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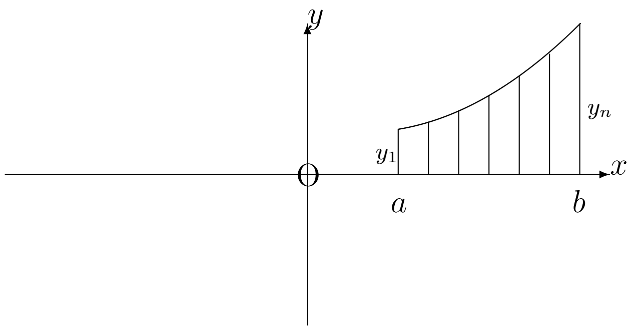
UNIT 13.2 - INTEGRATION APPLICATIONS 2

MEAN AND ROOT MEAN SQUARE VALUES

13.2.1 MEAN VALUES



On the curve whose equation is

$$y = f(x),$$

suppose that $y_1, y_2, y_3, ..., y_n$ are the y-coordinates which correspond to n different x-coordinates, $a = x_1, x_2, x_3, ..., x_n = b$.

The average (that is, the arithmetic mean) of these n y-coordinates is

$$\frac{y_1 + y_2 + y_3 + ... + y_n}{n}.$$

But now suppose that we wished to determine the average (arithmetic mean) of **all** the y-coordinates, from $x = a$ to $x = b$ on the curve whose equation is $y = f(x)$.

We could make a reasonable approximation by taking a very **large** number, n, of y-coordinates separated in the x-direction by very **small** distances. If these distances are typically represented by δx, then the required mean value could be written

$$\frac{y_1\delta x + y_2\delta x + y_3\delta x + ... + y_n\delta x}{n\delta x},$$

in which the denominator is equivalent to $(b - a + \delta x)$, since there are only $n - 1$ spaces between the n y-coordinates.

Allowing the number of y-coordinates to increase indefinitely, δx will to tend to zero and we obtain the formula for the **"Mean Value"** in the form

$$\text{M.V.} = \frac{1}{b-a} \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\delta x.$$

That is,

$$\text{M.V.} = \frac{1}{b-a} \int_a^b f(x) \text{ d}x.$$

Note:
In cases where the definite integral in this formula represents the area between the curve and the x-axis, the Mean Value provides the height of a rectangle, with base $b - a$, having the same area as that represented by the definite integral.

EXAMPLE

Determine the Mean Value of the function

$$f(x) \equiv x^2 - 5x$$

from $x = 1$ to $x = 4$.

Solution

The Mean Value is given by

$$\text{M.V.} = \frac{1}{4-1} \int_1^4 (x^2 - 5x) \text{ d}x = \frac{1}{3}\left[\frac{x^3}{3} - \frac{5x^2}{2}\right]_1^4 =$$

$$\frac{1}{3}\left[\left(\frac{64}{3} - 40\right) - \left(\frac{1}{3} - \frac{5}{2}\right)\right] = -\frac{33}{2}.$$

13.2.2 ROOT MEAN SQUARE VALUES

It is sometimes convenient to use an alternative kind of average for the values of a function, $f(x)$, between $x = a$ and $x = b$.

The **"Root Mean Square Value"** provides a measure of "central tendency" for the **numerical** values of $f(x)$ and is defined to be the square root of the Mean Value of $f(x)$ from $x = a$ to $x = b$.

Hence,

$$\text{R.M.S.V.} = \sqrt{\frac{1}{b-a}\int_a^b \ [f(x)]^2 \ \mathrm{d}x}.$$

EXAMPLE

Determine the Root Mean Square Value of the function, $f(x) \equiv x^2 - 5$, from $x = 1$ to $x = 3$.

Solution

The Root Mean Square Value is given by

$$\text{R.M.S.V.} = \sqrt{\frac{1}{3-1}\int_1^3 \ (x^2-5)^2 \ \mathrm{d}x}.$$

Temporarily ignoring the square root, we obtain the **"Mean Square Value"**,

$$\text{M.S.V.} = \frac{1}{2}\int_1^3 \ (x^4 - 10x^2 + 25) \ \mathrm{d}x$$

$$= \frac{1}{2}\left[\frac{x^5}{5} - \frac{10x^3}{3} + 25x\right]_1^3 = \frac{1}{2}\left[\left(\frac{243}{5} - \frac{270}{3} + 75\right) - \left(\frac{1}{5} - \frac{10}{3} + 25\right)\right] = \frac{176}{30}.$$

Thus,

$$\text{R.M.S.V.} = \sqrt{\frac{176}{30}} \simeq 2.422$$

13.2.3 EXERCISES

1. (a) Determine the Mean Value of the function, $(x-1)(x-2)$, from $x=1$ to $x=2$;
   (b) Determine, correct to three significant figures, the Mean Value of the function, $\frac{1}{2x+5}$, from $x=3$ to $x=5$;
   (c) Determine the Mean Value of the function, $\sin 2t$, from $t=0$ to $t=\frac{\pi}{2}$;
   (d) Determine, correct to three places of decimals, the Mean Value of the function, e^{-x}, from $x=1$ to $x=5$;
   (e) Determine, correct to three significant figures, the mean value of the function, xe^{-2x}, from $x=0$ to $x=2$.
2. (a) Determine the Root Mean Square Value of the function, $3x+1$, from $x=-2$ to $x=2$;
   (b) Determine the Root Mean Square Value, of the function, e^x, from $x=0$ to $x=1$, correct to three decimal places;
   (c) Determine the Root Mean Square Value of the function, $\cos x$, from $x=\frac{\pi}{2}$ to $x=\pi$;
   (d) Determine the Root Mean Square Value of the function, $(4x-5)^{\frac{3}{2}}$, from $x=1.25$ to $x=1.5$.

13.2.4 ANSWERS TO EXERCISES

1. (a) $-\frac{1}{6}$;
   (b) 0.0775;
   (c) $\frac{2}{\pi}$;
   (d) -0.076;
   (e) 0.114
2. (a) $\sqrt{13} \simeq= 3.606$;
   (b) 1.787;
   (c) $\frac{1}{\sqrt{2}}$;
   (d) $\frac{1}{2}$.