"JUST THE MATHS"

UNIT NUMBER

13.16

INTEGRATION APPLICATIONS 16
(Centres of pressure)

by

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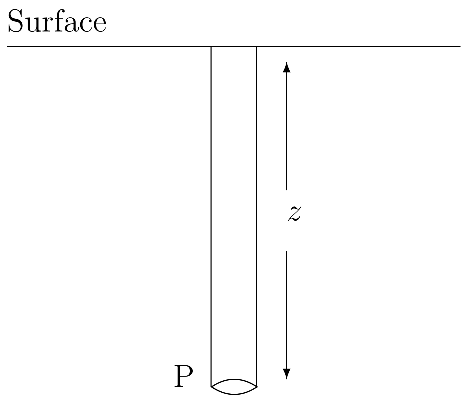
UNIT 13.16 - INTEGRATION APPLICATIONS 16

CENTRES OF PRESSURE

13.16.1 THE PRESSURE AT A POINT IN A LIQUID

In the following diagram, we consider the pressure in a liquid at a point, P, whose depth below the surface of the liquid is z.



Ignoring atmospheric pressure, the pressure, p, at P is measured as the thrust acting upon unit area and is due to the weight of the column of liquid with height z above it.

Hence,

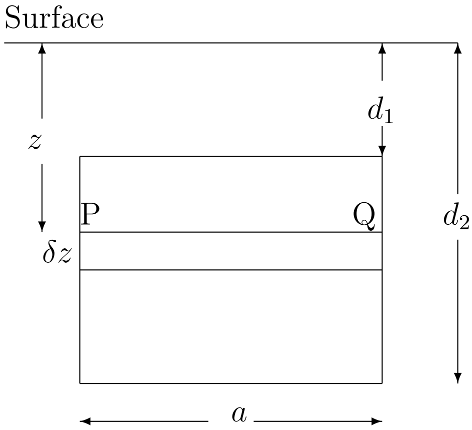
$$p = wz$$

where w is the weight, per unit volume, of the liquid.

Note:
The pressure at P is directly proportional to the depth of P below the surface; and we shall assume that the pressure acts equally in all directions at P.

13.16.2 THE PRESSURE ON AN IMMERSED PLATE

We now consider a rectangular plate, with dimensions a and $(d_2 - d_1)$, immersed vertically in a liquid as shown below.



For a thin strip, PQ, of width, δz, at a depth, z, below the surface of the liquid, the thrust on PQ will be the pressure at P multiplied by the area of the strip; that is, $wz \times a\delta z$.

The total thrust on the whole plate will therefore be

$$\sum_{z=d_1}^{z=d_2} waz\delta z.$$

Allowing δz to tend to zero, the total thrust becomes

$$\int_{d_1}^{d_2} waz \text{ d}z = \left[\frac{waz^2}{2}\right]_{d_1}^{d_2} = \frac{wa}{2}(d_2^2 - d_1^2).$$

This may be written

$$wa(d_2 - d_1)\left(\frac{d_2 + d_1}{2}\right),$$

where, in this form, $a(d_2 - d_1)$ is the area of the plate and $\frac{d_2+d_1}{2}$ is the depth of the centroid of the plate.

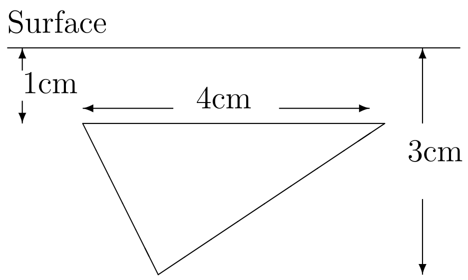
We conclude that

$$\text{Total Thrust} = \text{Area of Plate} \times \text{Pressure at the Centroid.}$$

Note:
It may be shown that this result holds whatever the shape of the plate is; and even when the plate is not vertical.

EXAMPLES

1. A triangular plate is immersed vertically in a liquid for which the weight per unit volume is w. The dimensions of the plate and its position in the liquid is shown in the following diagram:



Determine the total thrust on the plate as a multiple of w.

Solution

The area of the plate is given by

$$\text{Area} = \frac{1}{2} \times 4 \times 2 = 4\text{cm}^2$$

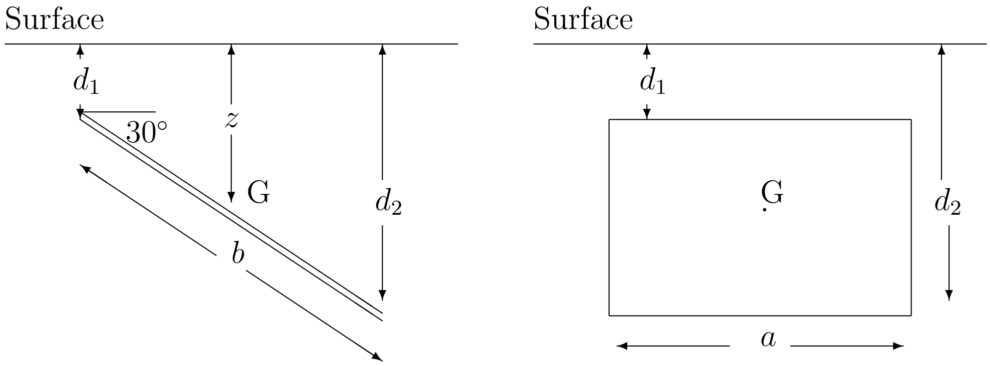
The centroid of the plate, which is at a distance from its horizontal side equal to one third of its perpendicular height, will lie at a depth of

$$\left(1 + \frac{1}{3} \times 2\right) \text{cms.} = \frac{5}{3}\text{cms.}$$

Hence, the pressure at the centroid is $\frac{5w}{3}$ and we conclude that

$$\text{Total thrust} = 4 \times \frac{5w}{3} = \frac{20w}{3}.$$

2. The following diagram shows a rectangular plate immersed in a liquid for which the weight per unit volume is w; and the plate is inclined at $30°$ to the horizontal:



Determine the total thrust on the plate as a multiple of w.

Solution

The depth, z, of the centroid, G, of the plate is given by

$$z = d_1 + \frac{b}{2}\sin 30° = d_1 + \frac{b}{4}.$$

Hence, the pressure, p, at G is given by

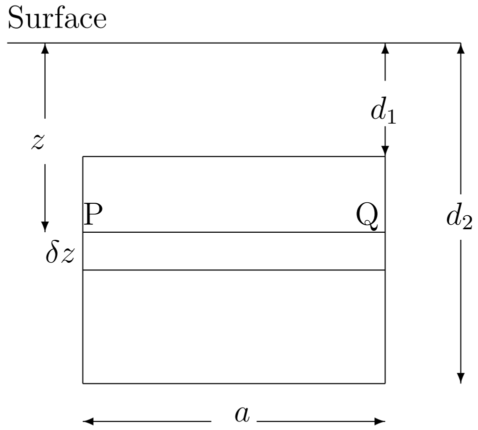
$$p = \left(d_1 + \frac{b}{4}\right) w;$$

and, since the area of the plate is ab, we obtain

$$\text{Total thrust} = ab\left(d_1 + \frac{b}{4}\right) w.$$

13.16.3 THE DEPTH OF THE CENTRE OF PRESSURE

In this section, we consider again an earlier diagram for a rectangular plate, immersed vertically in a liquid whose weight per unit volume is w.



We have already seen that the total thrust on the plate is

$$\int_{d_1}^{d_2} waz \ \mathrm{d}z = w\int_{d_1}^{d_2} az \ \mathrm{d}z$$

and is the resultant of varying thrusts, acting according to depth, at each level of the plate.

But, by taking first moments of these thrusts about the line in which the plane of the plate intersects the surface of the liquid, we may determine a particular depth at which the total thrust may be considered to act.

This depth is called **"the depth of the centre of pressure"**.

The Calculation

In the diagram, the thrust on the strip PQ is $waz\delta z$ and its first moment about the line in the surface is $waz^2\delta z$ so that the sum of the first moments on all such strips is given by

$$\sum_{z=d_1}^{z=d_2} waz^2\delta z = w\int_{d_1}^{d_2} az^2 \ \mathrm{d}z$$

where the definite integral is, in fact, the second moment of the plate about the line in the surface.

Next, we define the depth, C_p, of the centre of pressure to be such that

$$\text{Total thrust} \times C_p = \text{sum of first moments of strips like PQ.}$$

That is,

$$w\int_{d_1}^{d_2} az\ \mathrm{d}z \times C_p = w\int_{d_1}^{d_2} az^2\ \mathrm{d}z$$

and, hence,

$$C_p = \frac{\int_{d_1}^{d_2} az^2\ \mathrm{d}z}{\int_{d_1}^{d_2} az\ \mathrm{d}z},$$

which may be interpreted as

$$C_p = \frac{Ak^2}{A\overline{z}} = \frac{k^2}{\overline{z}},$$

where A is the area of the plate, k is the radius of gyration of the plate about the line in the surface of the liquid and $\overline{z}$ is the depth of the centroid of the plate.

Notes:

(i) It may be shown that the formula

$$C_p = \frac{k^2}{\overline{z}}$$

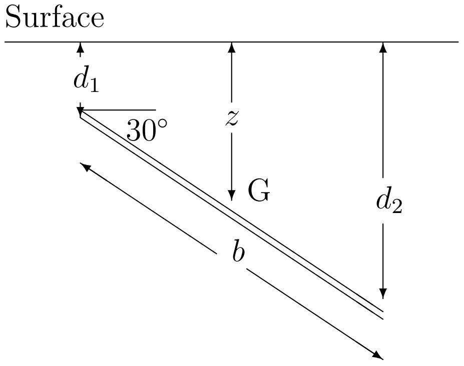
holds for any shape of plate immersed at any angle.

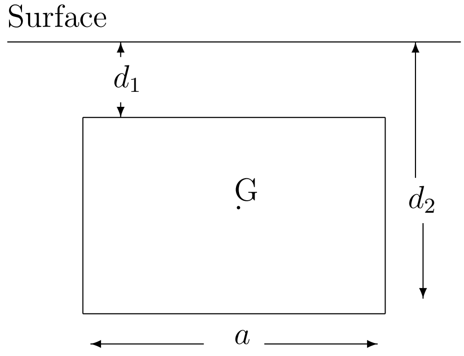
(ii) The phrase, "centre of pressure" suggests a particular point at which the total thrust is considered to act; but this is simply for convenience. The calculation is only for the depth of the centre of pressure.

EXAMPLE

Determine the depth of the centre of pressure for the second example of the previous section.

Solution





The depth of the centroid is

$$d_1 + \frac{b}{4}$$

and the square of the radius of gyration of the plate about an axis through the centroid, parallel to the side with length a is $\frac{a^2}{12}$.

Furthermore, the perpendicular distance between this axis and the line of intersection of the plane of the plate with the surface of the liquid is

$$\frac{b}{2} + \frac{d_1}{\sin 30°} = \frac{b}{2} + 2d_1.$$

Hence, the square of the radius of gyration of the plate about the line in the surface is

$$\frac{a^2}{12} + \left(\frac{b}{2} + 2d_1\right)^2,$$

using the Theorem of Parallel Axes.

Finally, the depth of the centre of pressure is given by

$$C_p = \frac{\frac{a^2}{12} + \left(\frac{b}{2} + 2d_1\right)^2}{d_1 + \frac{b}{4}}.$$

13.16.4 EXERCISES

1. A thin equilateral triangular plate is immersed vertically in a liquid for which the weight per unit volume is w, with one edge on the surface. If the length of each side is a, determine the total thrust on the plate.

2. A thin plate is bounded by the arc of a parabola and a straight line segment of length 1.2m perpendicular to the axis of symmetry of the parabola, this axis being of length 0.4m.

   If the plate is immersed vertically in a liquid with the straight edge on the surface, determine the total thrust on the plate in the form lw, where w is the weight per unit volume of the liquid and l is in decimals, correct to two places.

3. A thin rectangular plate, with sides of length 10cm and 20cm is immersed in a liquid so that the sides of length 10cm are horizontal and the sides of length 20cm are incline at 55° to the horizontal. If the uppermost side of the plate is at a depth of 13cm, determine the total thrust on then plate in the form lw, where w is the mass per unit volume of the liquid.

4. A thin circular plate, with diameter 0.5m is immersed vertically in a tank of liquid so that the uppermost point on its circumference is 2m below the surface. Determine the depth of the centre of pressure. correct to two places of decimals.

5. A thin plate is in the form of a trapezium with parallel sides of length 1m and 2.5m, a distance of 0.75m apart, and the remaining two sides inclined equally to either one of the parallel sides.

   If the plate is immersed vertically in water with the side of length 2.5m on the surface, calculate the depth of the centre of pressure, correct to two places of decimals.

13.16.5 ANSWERS TO EXERCISES

1.

$$\text{Total \ thrust} = \frac{wa^3}{8}.$$

2.

$$\text{Total \ Thrust} = 5.12w.$$

3.

$$C_p \simeq 2.26\text{m}.$$

4.

$$C_p \simeq 0.46\text{m}.$$