"JUST THE MATHS"

UNIT NUMBER

13.15

INTEGRATION APPLICATIONS 15
(Second moments of a surface of revolution)

by

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UNIT 13.15 - INTEGRATION APPLICATIONS 15

SECOND MOMENTS OF A SURFACE OF REVOLUTION

13.15.1 INTRODUCTION

Suppose that C denotes an arc (with length s) in the xy-plane of cartesian co-ordinates, and suppose that δs is the length of a small element of this arc.

Then, for the surface obtained when the arc is rotated through 2π radians about the x-axis, the **"second moment"** about the x-axis, is given by

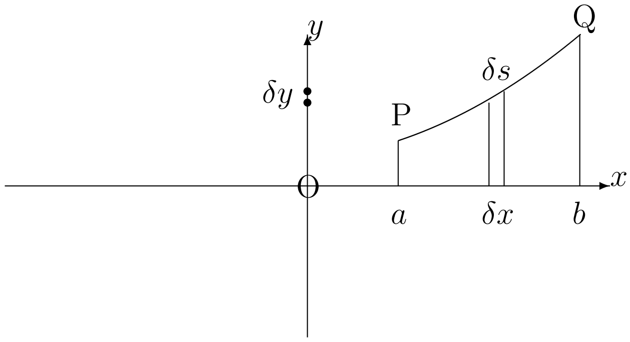
$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} y^2 . 2\pi y \delta s.$$

13.15.2 INTEGRATION FORMULAE FOR SECOND MOMENTS

(a) Let us consider an arc of the curve whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may be divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

From Pythagoras' Theorem,

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x$$

so that, for the surface of revolution of the arc about the x-axis, the second moment becomes

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi y^3 \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x = \int_a^b 2\pi y^3 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided that $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered. If not, then the above line needs to be prefixed by a negative sign.

From the technique of integration by substitution,

$$\int_a^b 2\pi y^3 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_{t_1}^{t_2} 2\pi y^3 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

We may conclude that the second moment about the plane through the origin, perpendicular to the x-axis, is given by

$$\text{second moment} = \pm \int_{t_1}^{t_2} 2\pi y^3 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\, \mathrm{d}t$$

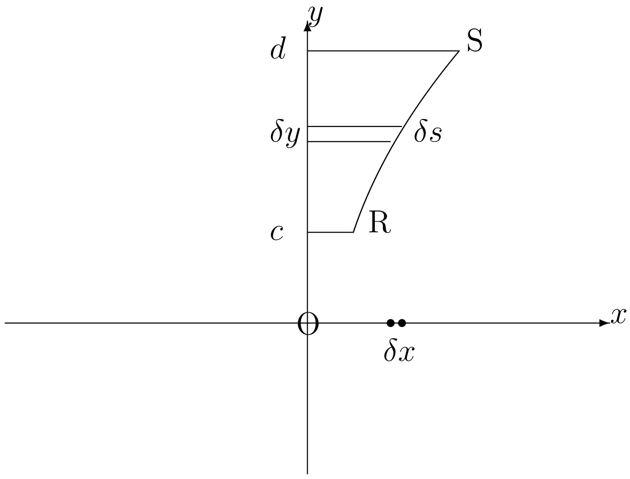
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in the previous section so that the second moment about the y-axis is given by

$$\int_c^d 2\pi x^3 \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\, \mathrm{d}y$$



Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$, then the second moment about the y-axis is given by

$$\text{second moment} = \pm \int_{t_1}^{t_2} 2\pi x^3 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative.

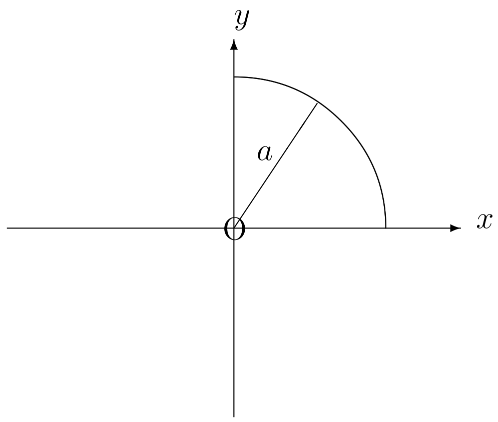
EXAMPLES

1. Determine the second moment about the x-axis of the hemispherical surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



Using implicit differentiation, we have

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and, hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

The second moment about the x-axis is therefore given by

$$\int_0^a 2\pi y^3 \sqrt{1 + \frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a 2\pi y^3 \sqrt{\frac{x^2 + y^2}{y^2}}\ \mathrm{d}x.$$

But $x^2 + y^2 = a^2$, and so the second moment becomes

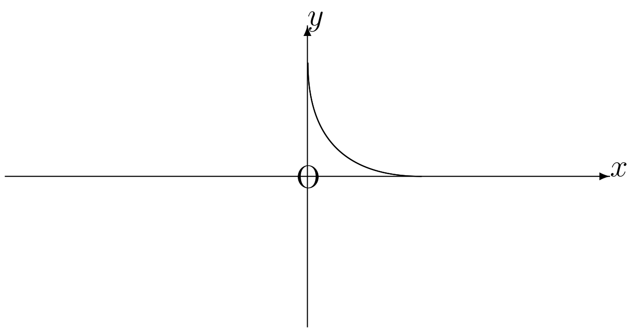
$$\int_0^a 2\pi a(a^2 - x^2)\ \mathrm{d}x = 2\pi a \left[a^2 x - \frac{x^3}{3}\right]_0^a = \frac{4\pi a^4}{3}.$$

2. Determine the second moment about the axis of revolution, when the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta$$

is rotated through 2π radians about (a) the x-axis and (b) the y-axis.

Solution



(a) Firstly, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta \sin\theta \ \text{ and } \ \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta \cos\theta.$$

Hence, the second moment about the x-axis is given by

$$-\int_{\frac{\pi}{2}}^0 2\pi y^3 \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta,$$

which, on using $\cos^2\theta + \sin^2\theta \equiv 1$, becomes

$$\int_0^{\frac{\pi}{2}} 2\pi a^3 \sin^{27}\theta . 3a\cos\theta\sin\theta \ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} 6\pi a^4 \sin^{28}\theta\cos\theta \ \mathrm{d}\theta$$

$$= 6\pi a^4 \int_0^{\frac{\pi}{2}} \sin^{28}\theta\cos\theta \ \mathrm{d}\theta,$$

which, by the methods of Unit 12.7 gives

$$6\pi a^4 \left[\frac{\sin^{29}\theta}{29}\right]_0^{\frac{\pi}{2}} = \frac{6\pi a^4}{29}.$$

(b) By symmetry, or by direct integration, the second moment about the y-axis is also $\frac{6\pi a^4}{29}$.

13.15.3 THE RADIUS OF GYRATION OF A SURFACE OF REVOLUTION

Having calculated the second moment of a surface of revolution about a specified axis, it is possible to determine a positive value, k, with the property that the second moment about the axis is given by Sk^2, where S is the total surface area of revolution.

We simply divide the value of the second moment by S in order to obtain the value of k^2 and hence the value of k.

The value of k is called the **"radius of gyration"** of the given arc about the given axis.

Note:
The radius of gyration effectively tries to concentrate the whole surface at a single point for the purposes of considering second moments; but, unlike a centroid, this point has no specific location.

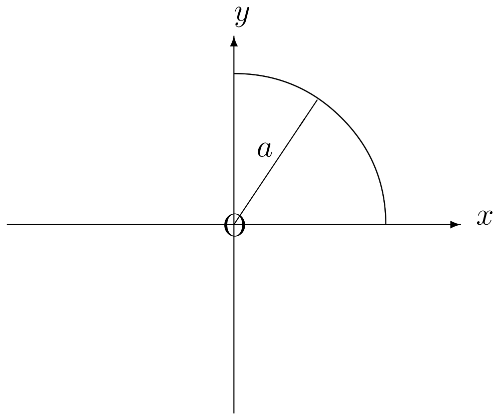
EXAMPLES

1. Determine the radius of gyration about the x-axis of the surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



From an Example 1 in section 13.15.2, we know that the second moment of the surface about the x-axis is equal to $\frac{4\pi a^4}{3}$.
Also, the total surface area is

$$\int_0^a 2\pi y\sqrt{1+\frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a 2\pi a\ \mathrm{d}x = 2\pi a^2,$$

which implies that

$$k^2 = \frac{4\pi a^4}{3} \times \frac{1}{2\pi a^2} \quad = \frac{2a^2}{3}.$$

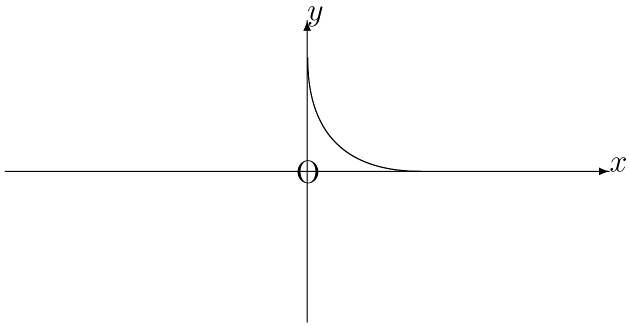
The radius of gyration is thus given by

$$k = a\sqrt{\frac{2}{3}}.$$

2. Determine the radius of gyration about the x-axis of the surface of revolution (about the x-axis) of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



We know from Example 2 in section 13.15.2, that that the second moment of the surface about the x-axis is equal to $\frac{6\pi a^4}{29}$.
Also, the total surface area is given by

$$-\int_{\frac{\pi}{2}}^{0} 2\pi a\sin^3\theta.3a\cos\theta\sin\theta \ \mathrm{d}\theta = \int_{0}^{\frac{\pi}{2}} 3a^2\sin^4\theta\cos\theta \ \mathrm{d}\theta = 3\pi a^2\left[\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}} = \frac{3\pi a^2}{5}.$$

Thus,

$$k^2 = \frac{6\pi a^4}{29} \times \frac{5}{3\pi a^2} = \frac{10a^2}{29}.$$

13.15.4 EXERCISES

1. Determine the second moment, about the x-axis, of the surface of revolution (about the x-axis) of the straight-line segment joining the origin to the point $(2, 3)$.
2. Determine the second moment about the x-axis, of the surface of revolution (about the x-axis) of the first quadrant arc of the curve whose equation is $y^2 = 4x$, lying between $x = 0$ and $x = 1$.
3. Determine, correct to two places of decimals, the second moment about the y-axis, of the surface of revolution (about the y-axis) of the first quadrant arc of the curve whose equation is $3y = x^3$, lying between $x = 1$ and $x = 2$.

4. Determine, correct to two places of decimals, the second moment, about the y-axis, of the surface of revolution (about the y-axis) of the arc of the circle given parametrically by

$$x = 2\cos t, \quad y = 2\sin t,$$

joining the point $(\sqrt{2}, \sqrt{2})$ to the point $(0, 2)$.

5. Determine the radius of gyration of a hollow right-circular cone with maximum radius, a, about its central axis.

6. For the curve whose equation is $9y^2 = x(3 - x)^2$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 - x}{2\sqrt{x}}.$$

Hence, show that the radius of gyration about the y-axis of the surface obtained when the first quadrant arch of this curve is rotated through 2π radians about the x-axis is 4, correct to the nearest whole number.

13.15.5 ANSWERS TO EXERCISES

1.

$$\frac{\pi 27\sqrt{13}}{2}.$$

2.

$$\frac{32\pi}{5}[4 - \sqrt{2}] \simeq 51.99$$

3.

$$70.44$$

4.

$$0.73$$

5.

$$k = \frac{a}{\sqrt{2}}.$$

6.

$$\text{Second moment} \simeq 139.92, \quad \text{surface area} \simeq 9.42$$