"JUST THE MATHS"

UNIT NUMBER

13.14

INTEGRATION APPLICATIONS 14
(Second moments of a volume (B))

by

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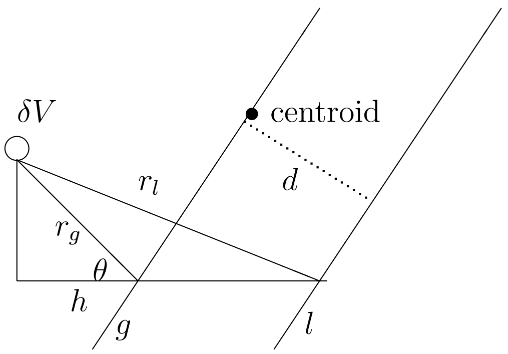
UNIT 13.14 - INTEGRATION APPLICATIONS 14

SECOND MOMENTS OF A VOLUME (B)

13.14.1 THE PARALLEL AXIS THEOREM

Suppose that M_g denotes the second moment of a given region, R, about an axis, g, through its centroid.

Suppose also that M_l denotes the second moment of R about an axis, l, which is parallel to the first axis and has a perpendicular distance of d from the first axis.



In the above **three-dimensional** diagram, we have

$$M_l = \sum_{\mathrm{R}} r_l^2 \delta V \text{ and } M_g = \sum_{\mathrm{R}} r_g^2 \delta V.$$

But, from the Cosine Rule,

$$r_l^2 = r_g^2 + d^2 - 2r_g d\cos(180^\circ - \theta) = r_g^2 + d^2 + 2r_g d\cos\theta.$$

Hence,

$$r_l^2 = r_g^2 + d^2 + 2dh$$

and so

$$\sum_R r_l^2 \delta V = \sum_R r_g^2 \delta V + \sum_R d^2 \delta V + 2d \sum_R h \delta V.$$

Finally, the expression

$$\sum_R h \delta V$$

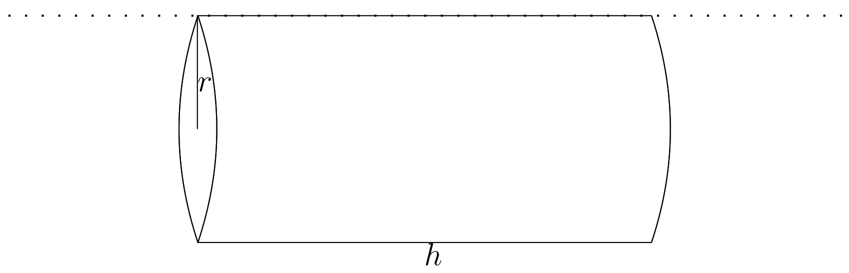
represents the first moment of R about a plane through the centroid, which is perpendicular to the plane containing l and g. Such first moment will be zero and hence,

$$M_l = M_g + Vd^2.$$

EXAMPLE

Determine the second moment of a solid right-circular cylinder about one of its generators (that is, a line in the surface, parallel to the central axis).

Solution



The second moment of the cylinder about the central axis was shown, in Unit 13.13, section 13.13.2, to be $\frac{\pi a^4 h}{2}$; and, since this axis and the generator are a distance a apart, the required second moment is given by

$$\frac{\pi a^4 h}{2} + (\pi a^2 h) a^2 = \frac{3\pi a^4 h}{2}.$$

13.14.2 THE RADIUS OF GYRATION OF A VOLUME

Having calculated the second moment of a three-dimensional region about a certain axis, it is possible to determine a positive value, k, with the property that the second moment about the axis is given by Vk^2, where V is the total volume of the region.

We simply divide the value of the second moment by V in order to obtain the value of k^2 and hence, the value of k.

The value of k is called the **"radius of gyration"** of the given region about the given axis.

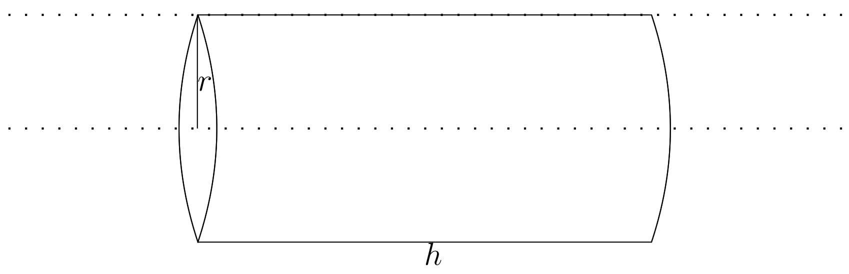
Note:

The radius of gyration effectively tries to concentrate the whole volume at a single point for the purposes of considering second moments; but, unlike a centroid, this point has no specific location.

EXAMPLES

1. Determine the radius of gyration of a solid right-circular cylinder with height, h, and radius, a, about (a) its own axis and (b) one of its generators.

   Solution

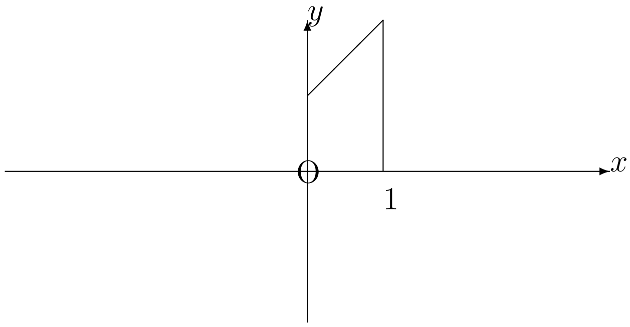
   

   Using earlier examples, together with the volume, $V = \pi a^2 h$, the required radii of gyration are (a) $\sqrt{\frac{\pi a^4 h}{2} \div \pi a^2 h} = \frac{a}{\sqrt{2}}$ and (b) $\sqrt{\frac{3\pi a^4 h}{2} \div \pi a^2 h} = a\sqrt{\frac{3}{2}}$.

2. Determine the radius of gyration of the volume of revolution about the x-axis of the region, bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the line whose equation is

$$y = x + 1.$$

Solution



From Unit 13.13, section 13.13.3, the second moment about the given axis is $\frac{31\pi}{10}$. The volume itself is given by

$$\int_0^1 \pi(x+1)^2 \, dx = \left[\pi\frac{(x+1)^3}{3}\right]_0^1 = \frac{7\pi}{3}.$$

Hence,

$$k^2 = \frac{31\pi}{10} \times \frac{3}{7\pi} = \frac{93}{70}.$$

That is,

$$k = \sqrt{\frac{93}{70}} \simeq 1.15$$

13.14.3 EXERCISES

1. Determine the radius of gyration of a hollow cylinder with internal radius, a, and external radius, b, about

   (a) its central axis;

   (b) a generator lying in its outer surface.

2. Determine the radius of gyration of a solid hemisphere, with radius a, about

   (a) its base-diameter;

   (b) an axis through its centroid, parallel to its base-diameter.

3. For a solid right-circular cylinder with height, h, and radius, a, determine the radius of gyration about

   (a) a diameter of one end;

   (b) an axis through the centroid, perpendicular to the axis of the cylinder.

4. For a solid right-circular cone with height, h, and base-radius, a, determine the radius of gyration about

   (a) the axis of the cone;

   (b) a line through the vertex, perpendicular to the axis of the cone;

   (c) a line through the centroid, perpendicular to the axis of the cone.

13.14.4 ANSWERS TO EXERCISES

1. (a)

$$\sqrt{\frac{a^2+b^2}{2}};$$

   (b)

$$\sqrt{\frac{3b^2+a^2}{2}}.$$

2. (a)

$$a\sqrt{\frac{2}{5}};$$

   (b)

$$a\sqrt{\frac{173}{320}}.$$

3. (a)

$$\sqrt{\frac{3a^2+4h^2}{12}};$$

   (b)

$$\sqrt{\frac{3a^2+7h^2}{12}}.$$

4. (a)

$$a\sqrt{\frac{3}{10}};$$

   (b)

$$\sqrt{\frac{3a^2}{20}+\frac{3h^2}{5}};$$

   (c)

$$\sqrt{\frac{3a^2}{20}+\frac{3h^2}{80}}.$$