"JUST THE MATHS"

UNIT NUMBER

13.13

INTEGRATION APPLICATIONS 13
(Second moments of a volume (A))

by

A.J.Hobson

UNIT 13.13 - INTEGRATION APPLICATIONS 13

SECOND MOMENTS OF A VOLUME (A)

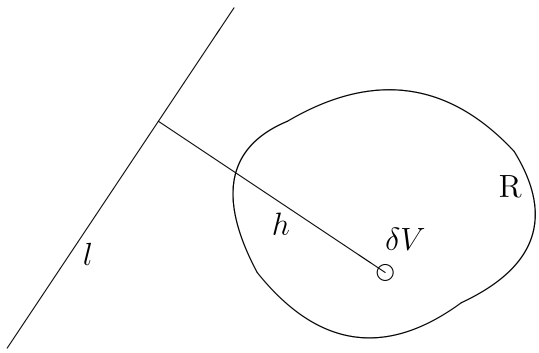
13.13.1 INTRODUCTION

Suppose that R denotes a region (with volume V) in space and suppose that δV is the volume of a small element of this region.

Then the **"second moment"** of R about a fixed line, l, is given by

$$\lim_{\delta V \to 0} \sum_{\mathrm{R}} h^2 \delta V,$$

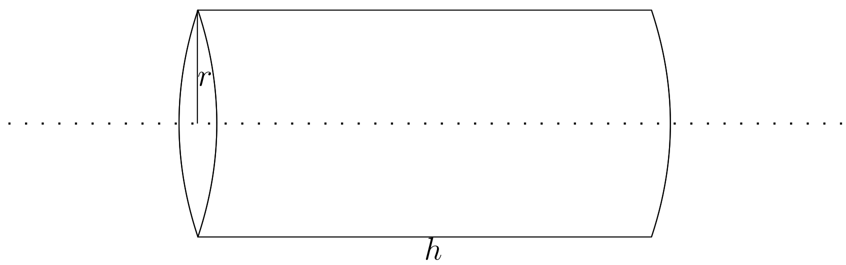
where h is the perpendicular distance from l of the element with volume δV.



EXAMPLE

Determine the second moment, about its own axis, of a solid right-circular cylinder with height, h, and radius, a.

Solution



In a thin cylindrical shell with internal radius, r, and thickness, δr, all of the elements of volume have the same perpendicular distance, r, from the axis of moments.

Hence the second moment of this shell will be the product of its volume and r^2; that is, $r^2(2\pi r h \delta r)$.

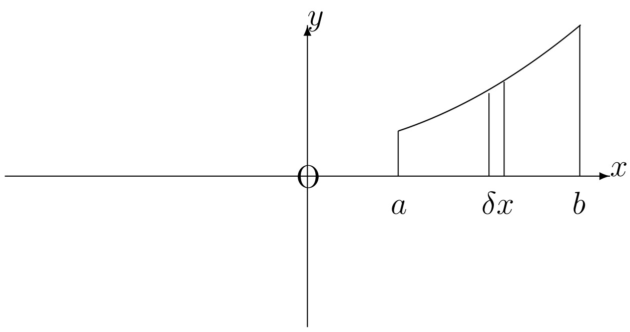
The total second moment is therefore given by

$$\lim_{\delta r \to 0} \sum_{r=0}^{r=a} r^2(2\pi r h \delta r) = \int_0^a 2\pi h r^3 \, \mathrm{d}r = \frac{\pi a^4 h}{2}.$$

13.13.2 THE SECOND MOMENT OF A VOLUME OF REVOLUTION ABOUT THE Y-AXIS

Let us consider a region in the first quadrant of the xy-plane, bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The volume of revolution of a narrow 'strip', of width, δx, and height, y, (parallel to the y-axis), is a cylindrical 'shell', of internal radius x, height, y, and thickness, δx.

Hence, from the example in the previous section, its second moment about the y-axis is $2\pi x^3 y \delta x$.

Thus, the total second moment about the y-axis is given by

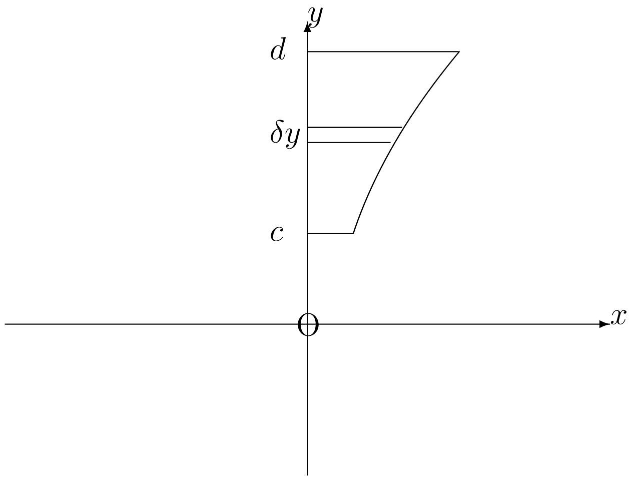
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi x^3 y \delta x = \int_a^b 2\pi x^3 y \, \mathrm{d}x.$$

Note:
Second moments about the x-axis will be discussed mainly in the next section of this Unit; but we note that, for the volume of revolution, about the x-axis, of a region in the first quadrant bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the second moment about the x-axis is given by

$$\int_c^d 2\pi y^3 x \, \mathrm{d}y.$$



EXAMPLE

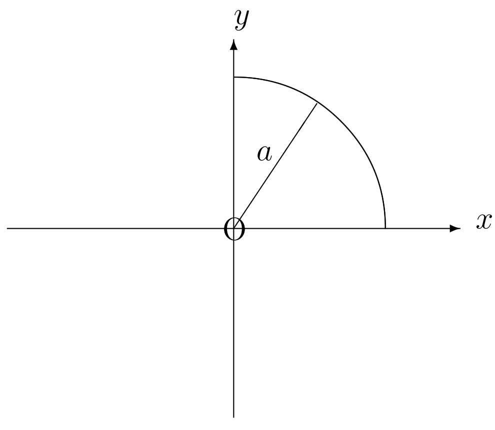
Determine the second moment, about a diameter, of a solid sphere with radius a.

Solution

We may consider, first, the volume of revolution about the y-axis of the region, bounded in the first quadrant, by the x-axis, the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2,$$

then double the result obtained.



The total second moment is given by

$$2\int_0^a 2\pi x^3\sqrt{a^2-x^2}\ \mathrm{d}x = 4\pi\int_0^{\frac{\pi}{2}} a^3\sin^3\theta.a\cos\theta.a\cos\theta\mathrm{d}\theta,$$

if we substitute $x = a\sin\theta$.

This simplifies to

$$4\pi a^5\int_0^{\frac{\pi}{2}} \sin^3\theta\ \cos^2\theta\mathrm{d}\theta = 4\pi\int_0^{\frac{\pi}{2}} \left(\cos^2\theta - \cos^4\theta\right)\sin\theta\mathrm{d}\theta,$$

if we make use of the trigonometric identity

$$\sin^2\theta \equiv 1 - \cos^2\theta.$$

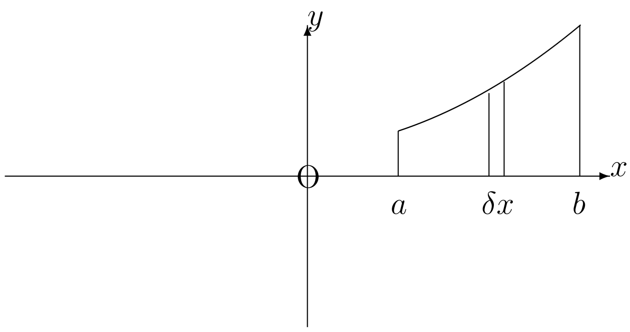
The total second moment is now given by

$$4\pi a^5\left[-\frac{\cos^3\theta}{3} + \frac{\cos^5\theta}{5}\right]_0^{\frac{\pi}{2}} = 4\pi a^5\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{8\pi a^5}{15}.$$

13.13.3 THE SECOND MOMENT OF A VOLUME OF REVOLUTION ABOUT THE X-AXIS

In the introduction to this Unit, a formula was established for the second moment of a solid right-circular cylinder about its own axis. This result may now be used to determine the second moment about the x-axis for the volume of revolution about this axis of a region enclosed in the first quadrant by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The volume of revolution about the x-axis of a narrow strip, of width δx and height y, is a cylindrical 'disc' whose second moment about the x-axis is $\frac{\pi y^4 \delta x}{2}$. Hence the second moment of the whole region about the x-axis is given by

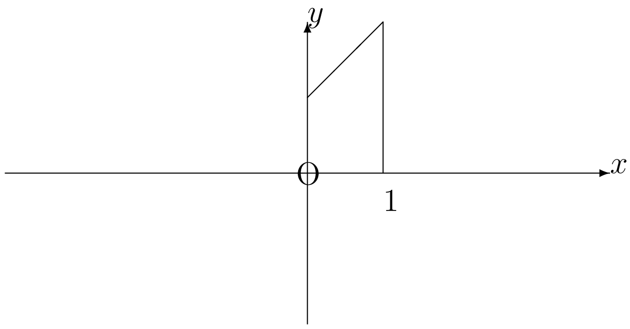
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \frac{\pi y^4}{2} \delta x = \int_a^b \frac{\pi y^4}{2} \, \mathrm{d}x.$$

EXAMPLE

Determine the second moment about the x-axis for the volume of revolution about this axis of the region bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the line whose equation is

$$y = x + 1.$$

Solution



$$\text{Second moment} = \int_0^1 \frac{\pi(x+1)^4}{2}\,\mathrm{d}x = \left[\pi\frac{(x+1)^4}{10}\right]_0^1 = \frac{31\pi}{10}.$$

Note:
The second moment of a volume about a certain axis is closely related to its **"moment of inertia"** about that axis. In fact, for a solid, with uniform density, ρ, the Moment of Inertia is ρ times the second moment of volume, since multiplication by ρ of elements of volume converts them into elements of mass.

13.13.4 EXERCISES

1. Determine the second moment about a diameter of a circular disc with small thickness, t, and radius, a.
2. Determine the second moment, about the axis specified, for the volume of revolution of each of the following regions of the xy-plane about this axis:
   (a) Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is
   $$y = 1 - 2x^2.$$
   Axis: The y-axis.
   (b) Bounded in the first quadrant by the x-axis and the curve whose equation is
   $$y^2 = \sin x.$$
   Axis: The x-axis.

(c) Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = e^{-2x}.$$

Axis: The x-axis

(d) Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = e^{-2x}.$$

Axis: The y-axis.

13.13.5 ANSWERS TO EXERCISES

1.

$$\frac{\pi a^4 t}{4}.$$

2. (a)

$$\frac{\pi}{24}.$$

(b)

$$\frac{\pi^2}{4}.$$

(c)

$$0.196, \text{ approximately.}$$

(d)

$$0.337, \text{ approximately.}$$