"JUST THE MATHS"

UNIT NUMBER

13.11

INTEGRATION APPLICATIONS 11
(Second moments of an area (A))

by

A.J.Hobson

UNIT 13.11 - INTEGRATION APPLICATIONS 11

SECOND MOMENTS OF AN AREA (A)

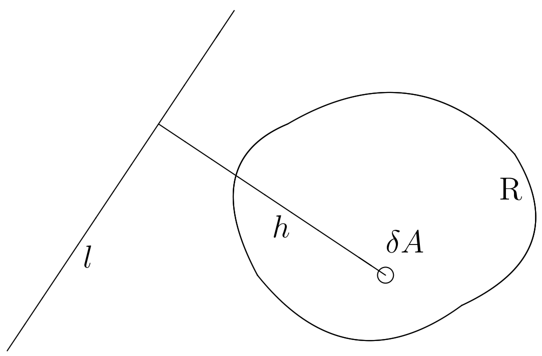
13.11.1 INTRODUCTION

Suppose that R denotes a region (with area A) of the xy-plane in cartesian co-ordinates, and suppose that δA is the area of a small element of this region.

Then the **"second moment"** of R about a fixed line, l, **not necessarily in the plane of** R, is given by

$$\lim_{\delta A \to 0} \sum_{\mathrm{R}} h^2 \delta A,$$

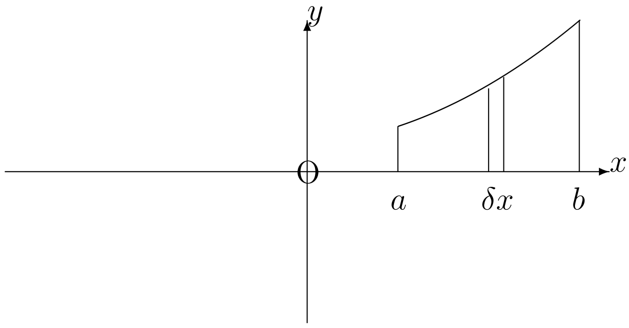
where h is the perpendicular distance from l of the element with area, δA.



13.11.2 THE SECOND MOMENT OF AN AREA ABOUT THE Y-AXIS

Let us consider a region in the first quadrant of the xy-plane bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The region may be divided up into small elements by using a network consisting of neighbouring lines parallel to the y-axis and neighbouring lines parallel to the x-axis.

But all of the elements in a narrow 'strip', of width δx and height y (parallel to the y-axis), have the same perpendicular distance, x, from the y-axis.

Hence the second moment of this strip about the y-axis is x^2 times the area of the strip; that is, $x^2(y\delta x)$, implying that the total second moment of the region about the y-axis is given by

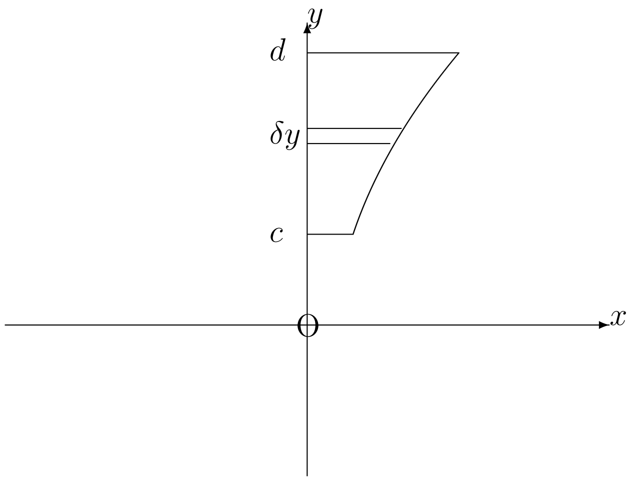
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} x^2 y \delta x = \int_a^b x^2 y \, \mathrm{d}x.$$

Note:
Second moments about the x-axis will be discussed mainly in the next section of this Unit; but we note that, for a region of the first quadrant, bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the second moment about the x-axis is given by

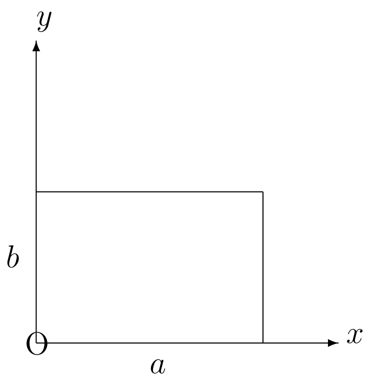
$$\int_c^d y^2 x \, \mathrm{d}y.$$



EXAMPLES

1. Determine the second moment of a rectangular region with sides of lengths, a and b, about the side of length b.

   Solution



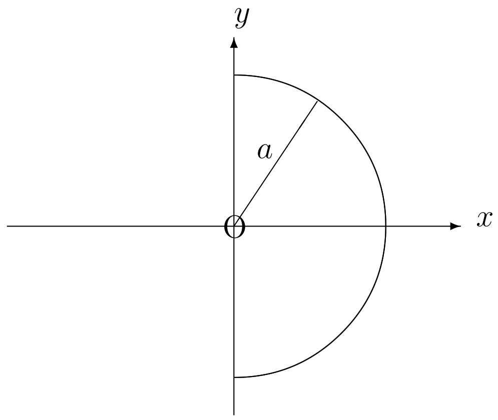
The second moment about the y-axis is given by

$$\int_0^a x^2 b \, \mathrm{d}x = \left[\frac{x^3 b}{3}\right]_0^a = \frac{1}{3}a^3 b.$$

2. Determine the second moment about the y-axis of the semi-circular region, bounded in the first and fourth quadrants, by the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$

Solution



Since there will be equal contributions from the upper and lower halves of the region, the second moment about the y-axis is given by

$$2\int_0^a x^2\sqrt{a^2 - x^2}\ \mathrm{d}x = 2\int_0^{\frac{\pi}{2}} a^2 \sin^2\theta . a\cos\theta . a\cos\theta \mathrm{d}\theta,$$

if we substitute $x = a\sin\theta$.

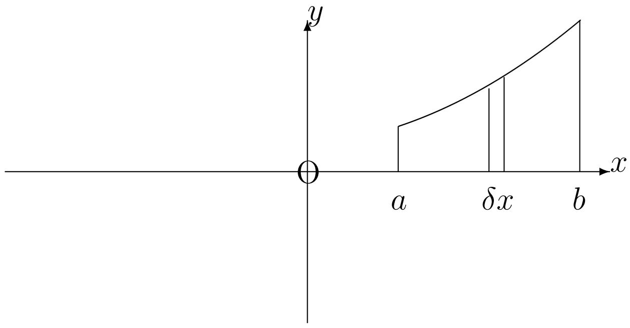
This simplifies to

$$2a^4\int_0^{\frac{\pi}{2}} \frac{\sin^2 2\theta}{4}\ \mathrm{d}\theta = \frac{a^4}{2}\int_0^{\frac{\pi}{2}} \frac{1 - \cos 4\theta}{2}\ \mathrm{d}\theta$$

$$= \frac{a^4}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi a^4}{8}.$$

13.11.3 THE SECOND MOMENT OF AN AREA ABOUT THE X-AXIS

In the first example of the previous section, a formula was established for the second moment of a rectangular region about one of its sides. This result may now be used to determine the second moment about the x-axis, of a region enclosed, in the first quadrant, by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is $y = f(x)$.



If a narrow strip of width δx and height y is regarded, approximately, as a rectangle, its second moment about the x-axis is $\frac{1}{3}y^3\delta x$. Hence the second moment of the whole region about the x-axis is given by

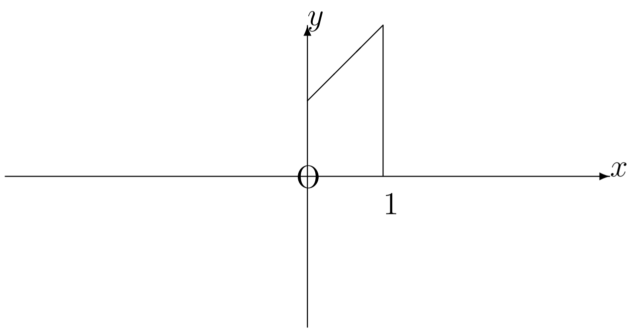
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \frac{1}{3}y^3\delta x = \int_a^b \frac{1}{3}y^3 \,\mathrm{d}x.$$

EXAMPLES

1. Determine the second moment about the x-axis of the region bounded, in the first quadrant, by the x-axis, the y-axis, the line $x = 1$ and the line whose equation is

$$y = x + 1.$$

Solution



$$\text{Second moment} = \int_0^1 \frac{1}{3}(x+1)^3 \, \mathrm{d}x$$

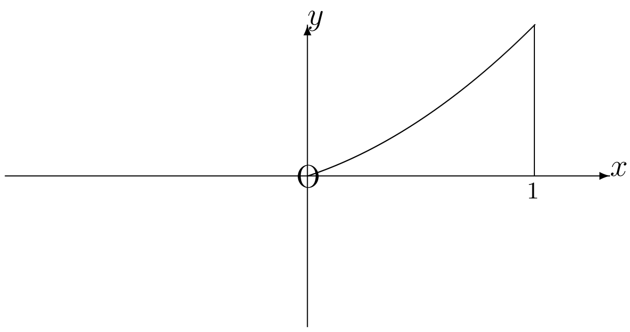
$$= \frac{1}{3}\int_0^1 (x^3+3x^2+3x+1) \, \mathrm{d}x = \frac{1}{3}\left[\frac{x^4}{4}+x^3++\frac{3x^2}{2}+x\right]_0^1$$

$$= \frac{1}{3}\left(\frac{1}{4}+1+\frac{3}{2}+1\right) = \frac{5}{4}.$$

2. Determine the second moment about the x-axis of the region, bounded in the first quadrant, by the x-axis, the y-axis, the line $x=1$ and the curve

$$y = xe^x.$$

Solution



$$\text{Second moment } = \int_0^1 \frac{1}{3}x^3 e^{3x}\ \mathrm{d}x$$

$$= \frac{1}{3}\left(\left[x^3\frac{e^{3x}}{3}\right]_0^1 - \int_0^1 x^2 e^{3x}\ \mathrm{d}x\right)$$

$$= \frac{1}{3}\left(\left[x^3\frac{e^{3x}}{3}\right]_0^1 - \left[x^2\frac{e^{3x}}{3}\right]_0^1 + \int_0^1 2x\frac{e^{3x}}{3}\ \mathrm{d}x\right)$$

$$= \frac{1}{3}\left(\left[x^3\frac{e^{3x}}{3}\right]_0^1 - \left[x^2\frac{e^{3x}}{3}\right]_0^1 + \frac{2xe^{3x}}{9} - \frac{2}{3}\int_0^1 \frac{e^{3x}}{3}\ \mathrm{d}x\right).$$

That is,

$$\frac{1}{3}\left[x^3\frac{e^{3x}}{3} - x^2\frac{e^{3x}}{3} + \frac{2xe^{3x}}{9} - \frac{2e^{3x}}{27}\right]_0^1 = \frac{4e^3+2}{81} \simeq 1.02$$

Note:
The Second Moment of an area about a certain axis is closely related to its **"moment of inertia"** about that axis. In fact, for a thin plate with uniform density, ρ, the moment of inertia is ρ times the second moment of area, since multiplication by ρ, of elements of area, converts them into elements of mass.

13.11.7 EXERCISES

Determine the second moment of each of the following regions of the xy-plane about the axis specified:

1. Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is

$$y = 1 - 2x^2.$$

   Axis: The y-axis.

2. Bounded in the first quadrant by the x-axis and the curve whose equation is

$$y = \sin x.$$

   Axis: The x-axis.

3. Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = e^{-2x}.$$

   Axis: The x-axis

4. Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = e^{-2x}.$$

   Axis: The y-axis.

13.11.8 ANSWERS TO EXERCISES

1.

$$\frac{\sqrt{2}}{30}.$$

2.

$$\frac{4}{9}.$$

3.

$$0.055, \text{ approximately.}$$

4.

$$0.083, \text{ approximately.}$$