"JUST THE MATHS"

UNIT NUMBER

13.10

INTEGRATION APPLICATIONS 10
(Second moments of an arc)

by

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UNIT 13.10 - INTEGRATION APPLICATIONS 10

SECOND MOMENTS OF AN ARC

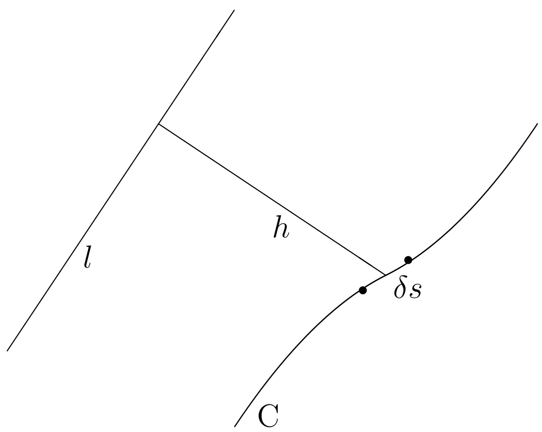
13.10.1 INTRODUCTION

Suppose that C denotes an arc (with length s) in the xy-plane of cartesian co-ordinates, and suppose that δs is the length of a small element of this arc.

Then the **"second moment"** of C about a fixed line, l, in the plane of C is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} h^2 \delta s,$$

where h is the perpendicular distance, from l, of the element with length δs.

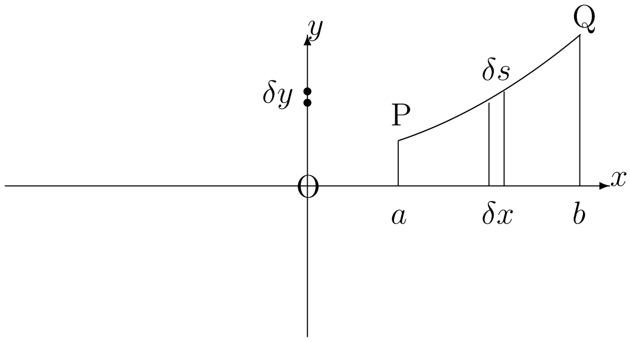


13.10.2 THE SECOND MOMENT OF AN ARC ABOUT THE Y-AXIS

Let us consider an arc of the curve whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may be divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

The second moment of each element about the y-axis is x^2 times the length of the element; that is, $x^2\delta s$, implying that the total second moment of the arc about the y-axis is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} x^2 \delta s.$$

But, by Pythagoras' Theorem,

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x,$$

so that the second moment of arc becomes

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} x^2 \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x = \int_a^b x^2 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, we may conclude that the second moment of the arc about the y-axis is given by

$$\pm\int_{t_1}^{t_2} x^2\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

13.10.3 THE SECOND MOMENT OF AN ARC ABOUT THE X-AXIS

(a) For an arc whose equation is

$$y = f(x),$$

contained between $x = a$ and $x = b$, the second moment about the x-axis will be

$$\int_a^b y^2\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, the second moment of the arc about the x-axis is given by

$$\pm\int_{t_1}^{t_2} y^2\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

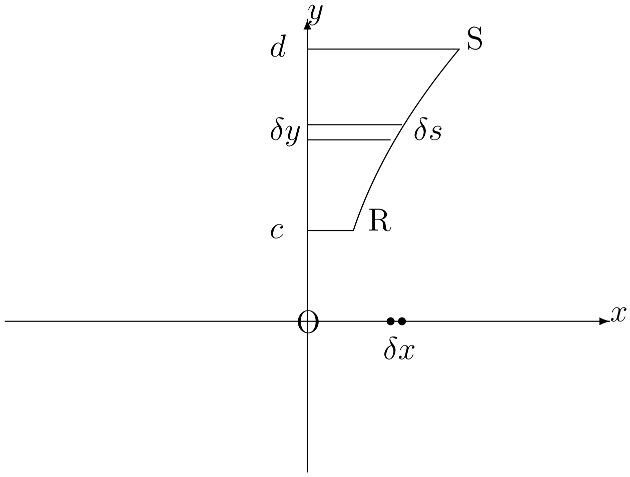
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in section 13.10.2 so that the second moment about the x-axis is given by

$$\int_c^d y^2 \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$



Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, we may conclude that the second moment of the arc about the x-axis is given by

$$\pm \int_{t_1}^{t_2} y^2 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative and where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$.

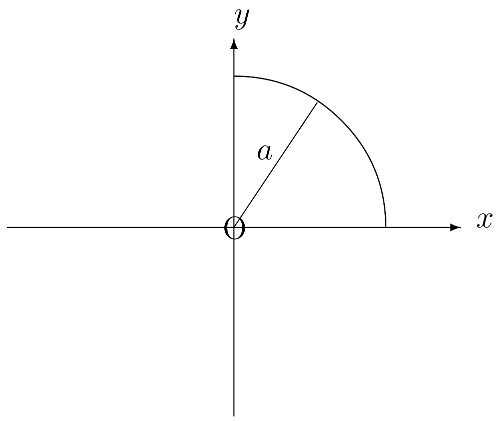
EXAMPLES

1. Determine the second moments about the x-axis and the y-axis of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



Using implicit differentiation, we have

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

The second moment about the y-axis is therefore given by

$$\int_0^a x^2\sqrt{1 + \frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a \frac{x^2}{y}\sqrt{x^2 + y^2}\ \mathrm{d}x.$$

But $x^2 + y^2 = a^2$ and, hence,

$$\text{second moment} = \int_0^a \frac{ax^2}{y}\mathrm{d}x.$$

Making the substitution $x = a\sin u$ gives

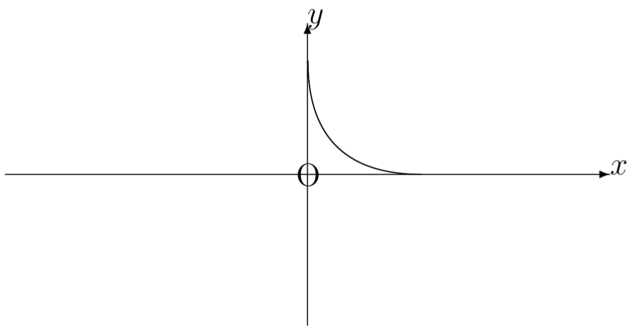
$$\text{second moment} = \int_0^{\frac{\pi}{2}} a^3\sin^2 u \; \mathrm{d}u = a^3 \int_0^{\frac{\pi}{2}} \frac{1-\cos 2u}{2} \; \mathrm{d}u = a^3 \left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi a^3}{4}.$$

By symmetry, the second moment about the x-axis will also be $\frac{\pi a^3}{4}$.

2. Determine the second moments about the x-axis and the y-axis of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



Firstly, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

Hence, the second moment about the y-axis is given by

$$-\int_{\frac{\pi}{2}}^{0} x^2\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta} \; \mathrm{d}\theta,$$

which, on using $\cos^2\theta + \sin^2\theta \equiv 1$, becomes

$$\int_0^{\frac{\pi}{2}} a^2\cos^6\theta . 3a\cos\theta\sin\theta \; \mathrm{d}\theta$$

$$= 3a^3 \int_0^{\frac{\pi}{2}} \cos^7\theta \sin\theta \ \mathrm{d}\theta$$

$$= 3a^2 \left[-\frac{\cos^8\theta}{8} \right]_0^{\frac{\pi}{2}} = \frac{3a^3}{8}.$$

Similarly, the second moment about the x-axis is given by

$$\int_0^{\frac{\pi}{2}} y^2 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2} \ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} a^2\sin^6\theta.(3a\cos\theta\sin\theta) \ \mathrm{d}\theta$$

$$= 3a^3 \int_0^{\frac{\pi}{2}} \sin^7\theta \cos\theta \ \mathrm{d}\theta = 3a^3 \left[\frac{\sin^8\theta}{8} \right]_0^{\frac{\pi}{2}} = \frac{3a^3}{8},$$

though, again, this second result could be deduced, by symmetry, from the first.

13.10.4 THE RADIUS OF GYRATION OF AN ARC

Having calculated the second moment of an arc about a certain axis it is possible to determine a positive value, k, with the property that the second moment about the axis is given by sk^2, where s is the total length of the arc.

We simply divide the value of the second moment by s in order to obtain the value of k^2 and, hence, the value of k.

The value of k is called the **"radius of gyration"** of the given arc about the given axis.

Note:
The radius of gyration effectively tries to concentrate the whole arc at a single point for the purposes of considering second moments; but, unlike a centroid, this point has no specific location.

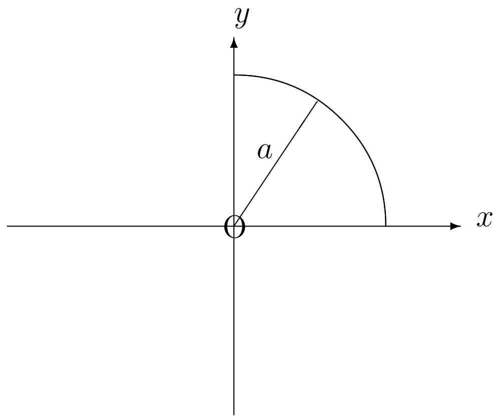
EXAMPLES

1. Determine the radius of gyration, about the y-axis, of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



From Example 1 in Section 13.10.3, we know that the Second Moment of the arc about the y-axis is equal to $\frac{\pi a^3}{4}$.

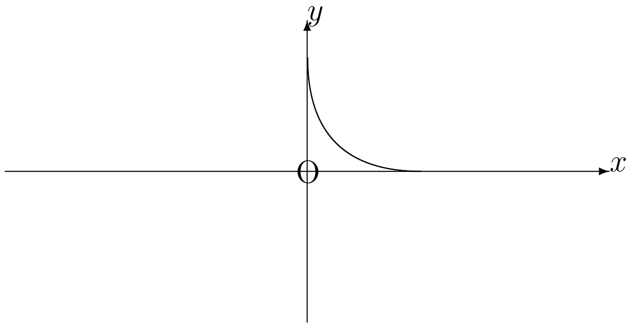
Also, the length of the arc is $\frac{\pi a}{2}$, which implies that the radius of gyration is

$$\sqrt{\frac{\pi a^3}{4} \times \frac{2}{\pi a}} = \frac{a}{\sqrt{2}}.$$

2. Determine the radius of gyration, about the y-axis, of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



From Example 2 in Section 13.10.3, we know that

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta$$

and that the second moment of the arc about the y-axis is equal to $\frac{3a^3}{8}$. Also, the length of the arc is given by

$$-\int_{\frac{\pi}{2}}^{a} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta.$$

This simplifies to

$$3a\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\ \mathrm{d}\theta = 3a\left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{3a}{2}.$$

Thus, the radius of gyration is

$$\sqrt{\frac{3a^3}{8} \times \frac{2}{3a}} = \frac{a}{2}.$$

13.10.5 EXERCISES

1. Determine the second moments about (a) the x-axis and (b) the y-axis of the straight line segment with equation

$$y = 2x + 1,$$

lying between $x = 0$ and $x = 3$.

2. Determine the second moment about the y-axis of the first-quadrant arc of the curve whose equation is

$$25y^2 = 4x^5,$$

lying between $x = 0$ and $x = 2$.

3. Determine, correct to two places of decimals, the second moment, about the x-axis, of the arc of the curve whose equation is

$$y = e^x,$$

lying between $x = 0.1$ and $x = 0.5$.

4. Given that

$$\int \sqrt{x^2 - 1}\ \mathrm{d}x = \frac{1}{2}\left(x\sqrt{x^2 - 1} - \ln(x + \sqrt{x^2 - 1})\right) + C,$$

determine, correct to two places of decimals, the second moment, about the x-axis, of the arc of the curve whose equation is

$$y^2 = 8x,$$

lying between $x = 0$ and $x = 1$.

5. Verify, using integration, that the radius of gyration, about the y-axis, of the straight line segment defined by the equation

$$y = 3x + 2,$$

from $x = 0$ to $x = 1$ is $\frac{1}{\sqrt{3}}$.

6. Determine the radius of gyration about the x-axis of the arc of the circle given parametrically by

$$x = 5\cos\theta, \quad y = 5\sin\theta,$$

from $\theta = 0$ to $\theta = \frac{\pi}{4}$.

7. For the curve whose equation is

$$9y^2 = x(3-x)^2,$$

show that

$$\frac{dy}{dx} = \frac{1-x}{2\sqrt{x}}.$$

Hence, determine, correct to three significant figures, the radius of gyration, about the y-axis, of the first quadrant arch of this curve.

13.10.6 ANSWERS TO EXERCISES

1.

$$\text{(a)} \ \frac{9\sqrt{5}}{2} \quad \text{(b)} \ 12\sqrt{5}.$$

2.

$$\frac{52}{9} \simeq 5.78$$

3.

$$1.29$$

4.

$$8.59$$

5.

$$\text{Second moment} = \frac{\sqrt{10}}{3} \quad \text{Length} = \sqrt{10}.$$

6.

$$k = \sqrt{\frac{125(\pi-2)}{10\pi}} \simeq 2.13$$

7.

$$k \simeq 1.68$$