"JUST THE MATHS"

UNIT NUMBER

13.1

INTEGRATION APPLICATIONS 1
(The area under a curve)

by

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UNIT 13.1 - INTEGRATION APPLICATIONS 1

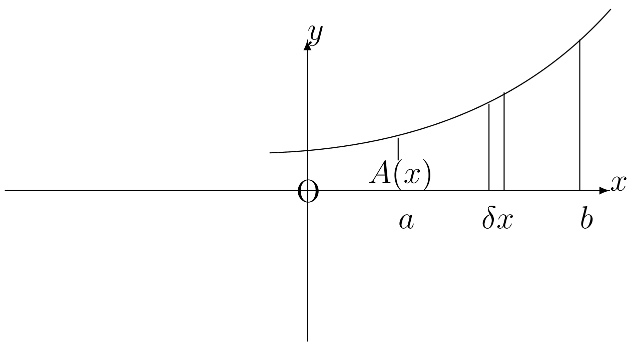
THE AREA UNDER A CURVE

13.1.1 THE ELEMENTARY FORMULA

We shall consider, here, a method of calculating the area contained between the x-axis of a cartesian co-ordinate system and the arc, from $x = a$ to $x = b$, of the curve whose equation is

$$y = f(x).$$

Suppose that $A(x)$ represents the area contained between the curve, the x-axis, the y-axis and the ordinate at some arbitrary value of x.



A small increase of δx in x will lead to a corresponding increase of δA in A approximating in area to that of a narrow rectangle whose width is δx and whose height is $f(x)$.

Thus,

$$\delta A \simeq f(x)\delta x,$$

which may be written

$$\frac{\delta A}{\delta x} \simeq f(x).$$

By allowing δx to tend to zero, the approximation disappears to give

$$\frac{\mathrm{d}A}{\mathrm{d}x} = f(x).$$

Hence, on integrating both sides with respect to x,

$$A(x) = \int f(x)\ \mathrm{d}x.$$

The constant of integration would need to be such that $A = 0$ when $x = 0$; but, in fact, we do not need to know the value of this constant because the required area, from $x = a$ to $x = b$, is given by

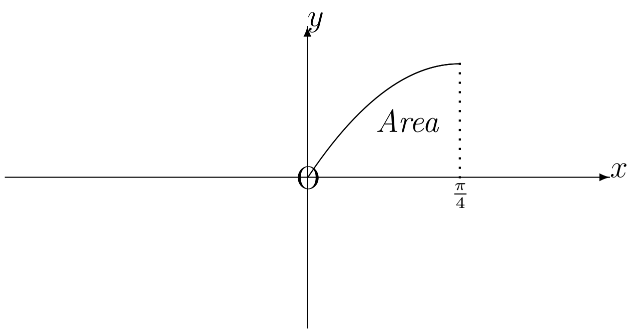
$$A(b) - A(a) = \int_a^b f(x)\ \mathrm{d}x.$$

EXAMPLES

1. Determine the area contained between the x-axis and the curve whose equation is $y = \sin 2x$, from $x = 0$ to $x = \frac{\pi}{4}$.

   Solution

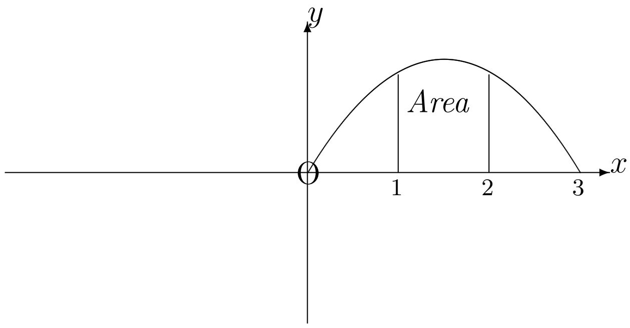
$$\int_0^{\frac{\pi}{4}} \sin 2x\ \mathrm{d}x = \left[-\frac{\cos 2x}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}.$$



2. Determine the area contained between the x-axis and the curve whose equation is $y = 3x - x^2$, from $x = 1$ to $x = 2$.

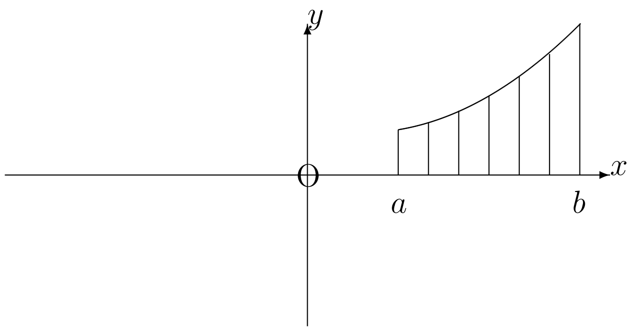
   Solution

$$\int_1^2 (3x - x^2)\, dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_1^2 = \left(6 - \frac{8}{3}\right) - \left(\frac{3}{2} - \frac{1}{3}\right) = \frac{13}{6}.$$



13.1.2 DEFINITE INTEGRATION AS A SUMMATION

Consider, now, the same area as in the previous section, but regarded (approximately) as the sum of a large number of narrow rectangles with typical width δx and typical height $f(x)$. The narrower the strips, the better will be the approximation.



Hence, we may state an alternative expression for the area from $x = a$ to $x = b$ in the form

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\delta x.$$

Since this new expression represents the same area as before, we may conclude that

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\delta x = \int_a^b f(x)\delta x.$$

Notes:

(i) The above result shows that an area which lies wholly **below** the x-axis will be **negative** and so care must be taken with curves which cross the x-axis between $x = a$ and $x = b$.

(ii) If c is any value of x between $x = a$ and $x = b$, the above result shows that

$$\int_a^b f(x)\ \mathrm{d}x = \int_a^c f(x)\ \mathrm{d}x + \int_c^b f(x)\ \mathrm{d}x.$$

(iii) To calculate the TOTAL area contained between the x-axis and a curve which crosses the x-axis between $x = a$ and $x = b$, account must be taken of any parts of the area which are negative.

(iv) It is usually a good idea to sketch the area under consideration before evaluating the appropriate definite integrals.

(v) It will be seen shortly that the formula obtained for definite integration as a summation has a wider field of application than simply the calculation of areas.

EXAMPLES

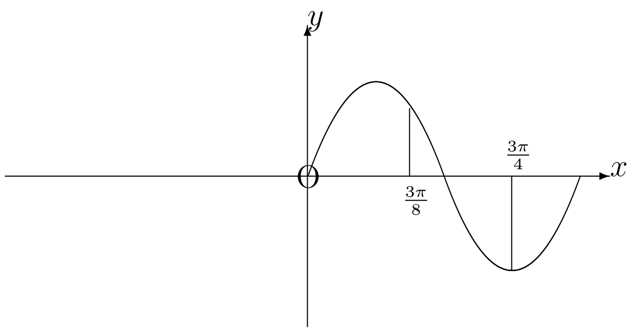
1. Determine the total area between the x-axis and the curve whose equation is $y = \sin 2x$, from $x = \frac{3\pi}{8}$ and $x = \frac{3\pi}{4}$.

   Solution

   $$\int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \sin 2x\ \mathrm{d}x - \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin 2x\ \mathrm{d}x.$$

   That is,

   $$\left[-\frac{\cos 2x}{2}\right]_{\frac{3\pi}{8}}^{\frac{\pi}{2}} - \left[-\frac{\cos 2x}{2}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}} = \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right) - \left(0 - \frac{1}{2}\right) = 1 - \frac{1}{2\sqrt{2}}.$$



2. Evaluate the definite integral,

$$\int_{\frac{3\pi}{8}}^{\frac{3\pi}{4}} \sin 2x \ \mathrm{d}x.$$

Solution

$$\int_{\frac{3\pi}{8}}^{\frac{3\pi}{4}} \sin 2x \ \mathrm{d}x = \left[-\frac{\cos 2x}{2}\right]_{\frac{3\pi}{8}}^{\frac{3\pi}{4}} = -\frac{1}{2\sqrt{2}}.$$

13.1.3 EXERCISES

1. Determine the areas bounded by the following curves and the x-axis between the ordinates $x = 1$ and $x = 3$:

   (a)
   $$y = 2x^2 + x + 1;$$
   (b)
   $$y = (1 - x)^2;$$
   (c)
   $$y = 2\sqrt{x}.$$

2. Sketch the curve whose equation is

   $$y = (1 - x)(2 + x)$$

   and determine the area contained between the x-axis and the portion of the curve above the x-axis.

3. To the nearest whole number, determine the area bounded between $x = 1$ and $x = 2$ by the curves whose equations are

$$y = 3e^{2x} \text{ and } y = 3e^{-x}.$$

4. Determine the area bounded between $x = 0$ and $x = \frac{\pi}{3}$ by the curves whose equations are

$$y = \sin x \text{ and } y = \sin 2x.$$

5. Determine the total area, from $x = 0$ to $x = \frac{3\pi}{10}$, contained between the x-axis and the curve whose equation is

$$y = \cos 5x.$$

13.1.4 ANSWERS TO EXERCISES

1. (a)

$$\frac{70}{3};$$

(b)

$$\frac{8}{3};$$

(c)

$$4\sqrt{3} - \frac{4}{3}.$$

2.

$$\frac{9}{2}.$$

3.

$$70.$$

4.

$$0.25$$

5.

$$\frac{2\sqrt{2} - 1}{5\sqrt{2}} - \simeq 0.259$$