"JUST THE MATHS"

UNIT NUMBER

12.9

INTEGRATION 9
(Reduction formulae)

by

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UNIT 12.9 - INTEGRATION 9

REDUCTION FORMULAE

INTRODUCTION

For certain integrals, both definite and indefinite, the function being integrated (that is, the "integrand") consists of a product of two functions, one of which involves an unspecified integer, say n. Using the method of integration by parts, it is sometimes possible to express such an integral in terms of a similar integral where n has been replaced by $(n-1)$, or sometimes $(n-2)$. The relationship between the two integrals is called a **"reduction formula"** and, by repeated application of this formula, the original integral may be determined in terms of n.

12.9.1 INDEFINITE INTEGRALS

The method will be illustrated by examples.

EXAMPLES

1. Obtain a reduction formula for the indefinite integral

$$I_n = \int x^n e^x \, \mathrm{d}x$$

and, hence, determine I_3.

Solution

Using integration by parts with $u = x^n$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = e^x$, we obtain

$$I_n = x^n e^x - \int e^x.nx^{n-1} \, \mathrm{d}x.$$

That is,

$$I_n = x^n e^x - nI_{n-1}.$$

Substituting $n = 3$,

$$I_3 = x^3 e^x - 3I_2,$$

where

$$I_2 = x^2 e^x - 2I_1$$

and

$$I_1 = xe^x - I_0.$$

But

$$I_0 = \int e^x \, \mathrm{d}x = e^x + \text{constant},$$

which leads us to the conclusion that

$$I_3 = x^3 e^x - 3\left[x^2 e^x - 2\left(xe^x - e^x\right)\right] + \text{constant}.$$

In other words,

$$I_3 = e^x \left[x^3 - 3x^2 + 6x - 6\right] + C,$$

where C is an arbitrary constant.

2. Obtain a reduction formula for the indefinite integral

$$I_n = \int x^n \cos x \, \mathrm{d}x$$

and, hence, determine I_2 and I_3.

Solution

Using integration by parts with $u = x^n$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$, we obtain

$$I_n = x^n \sin x - \int \sin x . nx^{n-1} \, \mathrm{d}x = x^n \sin x - n \int x^{n-1} \sin x \, \mathrm{d}x.$$

Using integration by parts in this last integral, with $u = x^{n-1}$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$, we obtain

$$I_n = x^n \sin x - n\left\{-x^{n-1}\cos x + \int \cos x.(n-1)x^{n-2} \, \mathrm{d}x\right\}.$$

That is,

$$I_n = x^n \sin x + nx^{n-1}\cos x - n(n-1)I_{n-2}.$$

Substituting $n = 2$,

$$I_2 = x^2 \sin x + 2x \cos x - 2I_0,$$

where

$$I_0 = \int \cos x \, \mathrm{d}x = \sin x + \text{constant}.$$

Hence,

$$I_2 = x^2 \sin x + 2x \cos x - 2 \sin x + C,$$

where C is an arbitrary constant.
Also, substituting $n = 3$,

$$I_3 = x^3 \sin x - 3x^2 \cos x - 3.2.I_1,$$

where

$$I_1 = \int x \cos x \, \mathrm{d}x = x \sin x + \cos x + \text{constant}.$$

Therefore,

$$I_3 = x^3 \sin x - 3x^2 \cos x - 6x \sin x - 6 \cos x + D,$$

where D is an arbitrary constant.

12.9.2 DEFINITE INTEGRALS

Integrals of the type encountered in the previous section may also include upper and lower limits of integration. The process of finding a reduction formula is virtually the same, except that the limits of integration are inserted where appropriate. Again, the method is illustrated by examples.

EXAMPLES

1. Obtain a reduction formula for the definite integral

$$I_n = \int_0^1 x^n e^x \, \mathrm{d}x$$

and, hence, determine I_3.

Solution
From the first example in section 12.9.1,

$$I_n = [x^n e^x]_0^1 - nI_{n-1} = e - nI_{n-1}.$$

Substituting $n = 3$,

$$I_3 = e - 3I_2,$$

where

$$I_2 = e - 2I_1$$

and

$$I_1 = e - I_0.$$

But

$$I_0 = \int_0^1 e^x \, \mathrm{d}x = e - 1,$$

which leads us to the conclusion that

$$I_3 = e - 3e + 6e - 6e + 6 = 6 - 2e.$$

2. Obtain a reduction formula for the definite integral

$$I_n = \int_0^{\pi} x^n \cos x \, \mathrm{d}x$$

and, hence, determine I_2 and I_3.
Solution
From the second example in section 12.9.1,

$$I_n = \left[x^n \sin x + nx^{n-1} \cos x\right]_0^{\pi} - n(n-1)I_{n-2} = -n\pi^{n-1} - n(n-1)I_{n-2}.$$

Substituting $n = 2$,

$$I_2 = -2\pi - 2I_0,$$

where

$$I_0 = \int_0^{\pi} \cos x \, \mathrm{d}x = [\sin x]_0^{\pi} = 0.$$

Hence,

$$I_2 = -2\pi.$$

Also, substituting $n = 3$,

$$I_3 = -3\pi^2 - 3.2.I_1,$$

where

$$I_1 = \int_0^\pi x \cos x \ \mathrm{d}x = [x \sin x + \cos x]_0^\pi = -2.$$

Therefore,

$$I_3 = -3\pi^2 + 12.$$

12.9.3 EXERCISES

1. Obtain a reduction formula for

$$I_n = \int \ x^n e^{2x} \ \mathrm{d}x$$

when $n \geq 1$ and, hence, determine I_3.

2. Obtain a reduction formula for

$$I_n = \int_0^1 \ x^n e^{2x} \mathrm{d}x$$

when $n \geq 1$ and, hence, evaluate I_4.

3. Obtain a reduction formula for

$$I_n = \int \ x^n \sin x \ \mathrm{d}x$$

when $n \geq 1$ and, hence, determine I_4.

4. Obtain a reduction formula for

$$I_n = \int_0^\pi \ x^n \sin x \ \mathrm{d}x$$

when $n \geq 1$ and, hence, evaluate I_3.

5. If

$$I_n = \int (\ln x)^n \, dx,$$

where $n \geq 1$, show that

$$I_n = x(\ln x)^n - nI_{n-1}$$

and, hence, determine I_3.

6. If

$$I_n = \int \left(x^2 + a^2\right)^n \, dx,$$

show that

$$I_n = \frac{1}{2n+1}\left[x\left(x^2 + a^2\right)^n + 2na^2 I_{n-1}\right].$$

Hint: Write $(x^2 + a^2)^n$ as $1.(x^2 + a^2)^n$.

12.9.4 ANSWERS TO EXERCISES

1.

$$I_n = \frac{1}{2}\left[x^n e^{2x} - nI_{n-1}\right],$$

giving

$$I_3 = \frac{e^{2x}}{8}\left[4x^3 - 6x^2 + 6x - 3\right] + C.$$

2.

$$I_n = \frac{1}{2}\left[e^2 - nI_{n-1}\right],$$

giving

$$I_4 = \frac{1}{4}\left[e^2 - 3\right].$$

3.

$$I_n = -x^n \cos x + nx^{n-1} \sin x - n(n-1)I_{n-2},$$

giving

$$I_4 = -x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x - 24x \sin x - 24 \cos x + C.$$

4.

$$I_n = \pi^n - n(n-1)I_{n-2},$$

giving

$$I_3 = \pi^3 - 6\pi.$$

5.

$$I_3 = x \left[\ln x)^3 - 3(\ln x)^2 + 6 \ln x - 6 \right] + C.$$