"JUST THE MATHS"

UNIT NUMBER

12.8

INTEGRATION 8
(The tangent substitutions)

by

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UNIT 12.8 - INTEGRATION 8

THE TANGENT SUBSTITUTIONS

There are two types of integral, involving sines and cosines, which require a special substitution using a tangent function. They are described as follows:

12.8.1 THE SUBSTITUTION $t = \tan x$

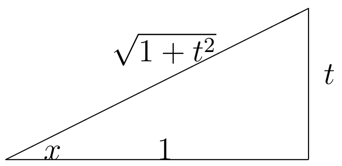
This substitution is used for integrals of the form

$$\int \frac{1}{a + b\sin^2 x + c\cos^2 x} \, dx,$$

where a, b and c are constants; though, in most exercises, at least one of these three constants will be zero.

A simple right-angled triangle will show that, if $t = \tan x$, then

$$\sin x \equiv \frac{t}{\sqrt{1+t^2}} \quad \text{and} \quad \cos x \equiv \frac{1}{\sqrt{1+t^2}}.$$



Furthermore,

$$\frac{dt}{dx} \equiv \sec^2 x \equiv 1 + t^2 \quad \text{so that} \quad \frac{dx}{dt} \equiv \frac{1}{1+t^2}.$$

EXAMPLES

1. Determine the indefinite integral

$$\int \frac{1}{4 - 3\sin^2 x} \, dx.$$

Solution

$$\int \frac{1}{4-3\sin^2 x}\,\mathrm{d}x$$

$$= \int \frac{1}{4-\frac{3t^2}{1+t^2}}.\frac{1}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{1}{4+t^2}\,\mathrm{d}t$$

$$= \frac{1}{2}\tan^{-1}\frac{t}{2} + C = \frac{1}{2}\tan^{-1}\left[\frac{\tan x}{2}\right] + C.$$

2. Determine the indefinite integral

$$\int \frac{1}{\sin^2 x + 9\cos^2 x}\,\mathrm{d}x.$$

Solution

$$\int \frac{1}{\sin^2 x + 9\cos^2 x}\,\mathrm{d}x$$

$$= \int \frac{1}{\frac{t^2}{1+t^2}+\frac{9}{1+t^2}}.\frac{1}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{1}{t^2+9}\,\mathrm{d}t$$

$$= \frac{1}{3}\tan^{-1}\frac{t}{3} + C = \frac{1}{3}\tan^{-1}\left[\frac{\tan x}{3}\right] + C.$$

12.8.2 THE SUBSTITUTION $t = \tan(x/2)$

This substitution is used for integrals of the form

$$\int \frac{1}{a + b\sin x + c\cos x}\,\mathrm{d}x,$$

where a, b and c are constants; though, in most exercises, one or more of these constants will be zero.

In order to make the substitution, we make the following observations:

(i)

$$\sin x \equiv 2\sin(x/2).\cos(x/2) \equiv 2\tan(x/2).\cos^2(x/2) \equiv \frac{2\tan(x/2)}{\sec^2(x/2)} \equiv \frac{2\tan(x/2)}{1+\tan^2(x/2)}.$$

Hence,

$$\sin x \equiv \frac{2t}{1+t^2}.$$

(ii)

$$\cos x \equiv \cos^2(x/2) - \sin^2(x/2) \equiv \cos^2(x/2)\left[1 - \tan^2(x/2)\right] \equiv \frac{1-\tan^2(x/2)}{\sec^2(x/2)} \equiv \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}.$$

Hence,

$$\cos x \equiv \frac{1-t^2}{1+t^2}.$$

(iii)

$$\frac{\mathrm{d}t}{\mathrm{d}x} \equiv \frac{1}{2}\sec^2(x/2) \equiv \frac{1}{2}\left[1+\tan^2(x/2)\right] \equiv \frac{1}{2}[1+t^2].$$

Hence,

$$\frac{\mathrm{d}x}{\mathrm{d}t} \equiv \frac{2}{1+t^2}.$$

EXAMPLES

1. Determine the indefinite integral

$$\int \frac{1}{1+\sin x}\,\mathrm{d}x$$

Solution

$$\int \frac{1}{1+\sin x}\,\mathrm{d}x$$

$$= \int \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{2}{1+t^2+2t}\,\mathrm{d}t$$

$$= \int \frac{2}{(1+t)^2}\,\mathrm{d}t$$

$$= -\frac{2}{1+t} + C = -\frac{2}{1+\tan(x/2)} + C.$$

2. Determine the indefinite integral

$$\int \frac{1}{4\cos x - 3\sin x}\,\mathrm{d}x.$$

Solution

$$\int \frac{1}{4\cos x - 3\sin x}\,\mathrm{d}x$$

$$= \int \frac{1}{4\frac{1-t^2}{1+t^2} - \frac{6t}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{2}{4-4t^2-6t}\,\mathrm{d}t = \int -\frac{1}{2t^2+3t-2}\,\mathrm{d}t$$

$$= \int -\frac{1}{(2t-1)(t+2)}\,\mathrm{d}t$$

$$= \int \frac{1}{5}\left[\frac{1}{t+2} - \frac{2}{2t-1}\right]\,\mathrm{d}t$$

$$= \frac{1}{5}[\ln(t+2) - \ln(2t-1)] + C = \frac{1}{5}\ln\left[\frac{\tan(x/2)+2}{2\tan(x/2)-1}\right] + C.$$

12.8.3 EXERCISES

1. Determine the indefinite integral

$$\int \frac{1}{4 + 12\cos^2 x}\,dx.$$

2. Evaluate the definite integral

$$\int_0^{\frac{\pi}{4}} \frac{1}{5\cos^2 x + 3\sin^2 x}\,dx.$$

3. Determine the indefinite integral

$$\int \frac{1}{5 + 3\cos x}\,dx.$$

4. Evaluate the definite integral

$$\int_3^{3.1} \frac{1}{12\sin x + 5\cos x}\,dx.$$

12.8.4 ANSWERS TO EXERCISES

1.

$$\frac{1}{4}\tan^{-1}\left[\frac{\tan x}{2}\right] + C.$$

2.

$$\left[\frac{1}{\sqrt{15}}\tan^{-1}\left(\sqrt{\frac{3}{5}}\tan x\right)\right]_0^{\frac{\pi}{4}} \simeq 0.1702$$

3.

$$\frac{1}{2}\tan^{-1}\left[\frac{\tan(x/2)}{2}\right] + C.$$

4.

$$\left[\frac{1}{13}[5\ln(5\tan(x/2) + 1) - \ln(\tan(x/2) - 5)\right]_3^{3.1} \simeq 0.348$$