"JUST THE MATHS"

UNIT NUMBER

12.7

INTEGRATION 7
(Further trigonometric functions)

by

A.J.Hobson

12.7.1 Products of sines and cosines
12.7.2 Powers of sines and cosines
12.7.3 Exercises
12.7.4 Answers to exercises

UNIT 12.7 - INTEGRATION 7 - FURTHER TRIGONOMETRIC FUNCTIONS

12.7.1 PRODUCTS OF SINES AND COSINES

In order to integrate the product of a sine and a cosine, or two cosines, or two sines, we may use one of the following trigonometric identities:

$$\text{sinAcosB} \equiv \frac{1}{2}\left[\sin(\text{A}+\text{B}) + \sin(\text{A}-\text{B})\right];$$

$$\text{cosAsinB} \equiv \frac{1}{2}\left[\sin(\text{A}+\text{B}) - \sin(\text{A}-\text{B})\right];$$

$$\text{cosAcosB} \equiv \frac{1}{2}\left[\cos(\text{A}+\text{B}) + \cos(\text{A}-\text{B})\right];$$

$$\text{sinAsinB} \equiv \frac{1}{2}\left[\cos(\text{A}-\text{B}) - \cos(\text{A}+\text{B})\right].$$

EXAMPLES

1. Determine the indefinite integral

$$\int \sin 2x \cos 5x \ \mathrm{d}x.$$

Solution

$$\int \sin 2x \cos 5x \ \mathrm{d}x = \frac{1}{2}\int \left[\sin 7x - \sin 3x\right] \ \mathrm{d}x$$

$$= -\frac{\cos 7x}{14} + \frac{\cos 3x}{6} + C.$$

2. Determine the indefinite integral

$$\int \sin 3x \sin x \ \mathrm{d}x.$$

Solution

$$\int \sin 3x \sin x \ \mathrm{d}x = \frac{1}{2}\int \left[\cos 2x - \cos 4x\right] \ \mathrm{d}x$$

$$= \frac{\sin 2x}{4} - \frac{\sin 4x}{8} + C.$$

12.7.2 POWERS OF SINES AND COSINES

In this section, we consider the two integrals,

$$\int \sin^n x \, dx \quad \text{and} \quad \int \cos^n \, dx,$$

where n is a positive integer.

(a) The Complex Number Method

A single method which will cover both of the above integrals requires us to use the methods of Unit 6.5 in order to express $\cos^n x$ and $\sin^n x$ as a sum of whole multiples of sines or cosines of whole multiples of x.

EXAMPLE

Determine the indefinite integral

$$\int \sin^4 x \, dx.$$

Solution

By the complex number method,

$$\sin^4 x \equiv \frac{1}{8}[\cos 4x - 4\cos 2x + 3].$$

The Working:

$$j^4 2^4 \sin^4 x \equiv \left(z - \frac{1}{z}\right)^4,$$

where $z \equiv \cos x + j \sin x$.

That is,

$$16\sin^4 x \equiv z^4 - 4z^3.\frac{1}{z} + 6z^2.\left(\frac{1}{z}\right)^2 - 4z.\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4;$$

or, after cancelling common factors,

$$16\sin^4 x \equiv z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4} \equiv z^4 + \frac{1}{z^4} - 4\left(z^2 + \frac{1}{z^2}\right) + 6,$$

which gives

$$16\sin^4 x \equiv 2\cos 4x - 8\cos 2x + 6,$$

or

$$\sin^4 x \equiv \frac{1}{8}[\cos 4x - 4\cos 2x + 3].$$

Hence,

$$\int \sin^4 x \, \mathrm{d}x = \frac{1}{8}\left[\frac{\sin 4x}{4} - 4\frac{\sin 2x}{2} + 3x\right] + C$$

$$= \frac{1}{32}[\sin 4x - 8\sin 2x + 12x] + C.$$

(b) Odd Powers of Sines and Cosines

The following method uses the facts that

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \cos x \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}[\cos x] = \sin x.$$

We illustrate with examples in which use is made of the trigonometric identity

$$\cos^2 A + \sin^2 A \equiv 1.$$

EXAMPLES

1. Determine the indefinite integral

$$\int \sin^3 x \, \mathrm{d}x.$$

Solution

$$\int \sin^3 x \; \mathrm{d}x = \int \sin^2 x . \sin x \; \mathrm{d}x.$$

That is,

$$\int \sin^3 x \mathrm{d}x = \int \left(1 - \cos^2 x\right) \sin x \; \mathrm{d}x$$

$$= \int \left(\sin x - \cos^2 x . \sin x\right) \; \mathrm{d}x$$

$$= -\cos x + \frac{\cos^3 x}{3} + C.$$

2. Determine the indefinite integral

$$\int \cos^7 x \; \mathrm{d}x.$$

Solution

$$\int \cos^7 x \; \mathrm{d}x = \int \cos^6 x . \cos x \; \mathrm{d}x.$$

That is,

$$\int \cos^7 x \; \mathrm{d}x = \int \left(1 - \sin^2 x\right)^3 . \cos x \; \mathrm{d}x$$

$$= \int \left(1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x\right) . \cos x \; \mathrm{d}x$$

$$= \sin x - \sin^3 x + 3 . \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.$$

(c) Even Powers of Sines and Cosines

The method illustrated here becomes tedious if the even power is higher than 4. In such cases, it is best to use the complex number method in paragraph (a) above.

In the examples which follow, we shall need the trigonometric identity

$$\cos 2A \equiv 1 - 2\sin^2 A \equiv 2\cos^2 A - 1.$$

EXAMPLES

1. Determine the indefinite integral

$$\int \sin^2 x \ \mathrm{d}x.$$

Solution

$$\int \sin^2 x \ \mathrm{d}x = \int \frac{1}{2}(1 - \cos 2x) \ \mathrm{d}x = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

2. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^2 x \ \mathrm{d}x.$$

Solution

$$\int_0^{\frac{\pi}{2}} \cos^2 x \ \mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2x) \ \mathrm{d}x$$

$$= \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

3. Determine the indefinite integral

$$\int \cos^4 x \ \mathrm{d}x.$$

Solution

$$\int \cos^4 x \ \mathrm{d}x = \int \left[\cos^2 x\right]^2 \ \mathrm{d}x = \int \left[\frac{1}{2}(1 + \cos 2x)\right]^2 \ \mathrm{d}x.$$

That is,

$$\int \cos^4 x \ \mathrm{d}x = \int \frac{1}{4}\left(1 + 2\cos 2x + \cos^2 2x\right) \ \mathrm{d}x$$

$$= \int \frac{1}{4}\left(1 + 2\cos 2x + \frac{1}{2}[1 + \cos 4x]\right) \ \mathrm{d}x$$

$$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{32} + C$$

$$= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C.$$

12.7.3 EXERCISES

1. Determine the indefinite integral

$$\int \cos x \cos 3x \ \mathrm{d}x.$$

2. Evaluate the definite integral

$$\int_0^{\frac{\pi}{3}} \cos 4x \sin 2x \ \mathrm{d}x.$$

3. Determine the following indefinite integrals:

   (a)

$$\int \sin^5 x \ \mathrm{d}x;$$

   (b)

$$\int \cos^3 x \ \mathrm{d}x.$$

4. Evaluate the following definite integrals:

   (a)

$$\int_0^{\frac{\pi}{8}} \sin^4 x \ \mathrm{d}x;$$

   (b)

$$\int_0^{\frac{\pi}{2}} \cos^6 x \ \mathrm{d}x.$$

12.7.4 ANSWERS TO EXERCISES

1.

$$\frac{\sin 4x}{8} + \frac{\sin 2x}{4} + C.$$

2.

$$-\frac{\sqrt{3}}{4} \simeq -0.433$$

3. (a)

$$-\frac{\cos^5 x}{5} + 2\frac{\cos^3 x}{3} - \cos x + C,$$

or

$$\frac{1}{16}\left[-\frac{\cos 5x}{5} + \frac{5\cos 3x}{3} - 10\cos x\right] + C \quad \text{by complex numbers;}$$

(b)

$$\sin x - \frac{\sin^3 x}{3} + C,$$

or

$$\frac{1}{4}\left[\frac{\sin 3x}{3} - 3\sin x\right] + C \quad \text{by complex numbers.}$$

4. (a)

$$1.735 \times 10^{-3} \text{ approx;}$$

(b)

$$-1.$$