"JUST THE MATHS"

UNIT NUMBER

12.6

INTEGRATION 6
(Integration by partial fractions)

by

A.J.Hobson

UNIT 12.6 - INTEGRATION 6
INTEGRATION BY PARTIAL FRACTIONS

12.6.1 INTRODUCTION AND ILLUSTRATIONS

If the ratio of two polynomials, whose denominator has been factorised, is expressed as a sum of partial fractions, each partial fraction will be of a type whose integral can be determined by the methods of preceeding sections of this chapter.

The following summary of results will cover most elementary problems involving partial fractions:

RESULTS

1.

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln(ax+b) + C.$$

2.

$$\int \frac{1}{(ax+b)^n}\,\mathrm{d}x = \frac{1}{a}.\frac{(ax+b)^{-n+1}}{-n+1} + C \quad \text{provided} \quad n \neq 1.$$

3.

$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

4.

$$\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C,$$

or

$$\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C \quad \text{when} \quad |x| < a,$$

and

$$\frac{1}{2a}\ln\left(\frac{x+a}{x-a}\right) + C \quad \text{when} \quad |x| > a.$$

5.

$$\int \frac{2ax+b}{ax^2+bx+c}\,\mathrm{d}x = \ln(ax^2+bx+c) + C.$$

ILLUSTRATIONS

We use some of the results of examples on partial fractions in Unit 1.8

1.

$$\int \frac{7x+8}{(2x+3)(x-1)}\,\mathrm{d}x = \int \left[\frac{1}{2x+3} + \frac{3}{x-1}\right]\,\mathrm{d}x$$

$$= \frac{1}{2}\ln(2x+3) + 3\ln(x-1) + C.$$

2.

$$\int_6^8 \frac{3x^2+9}{(x-5)(x^2+2x+7)}\,\mathrm{d}x = \int_6^8 \left[\frac{2}{x-5} + \frac{x+1}{x^2+2x+7}\right]\,\mathrm{d}x$$

$$= \left[2\ln(x-5) + \frac{1}{2}\ln(x^2+2x+7)\right]_6^8 \simeq 2.427$$

3.

$$\int \frac{9}{(x+1)^2(x-2)} = \int \left[\frac{-1}{x+1} - \frac{3}{(x+1)^2} + \frac{1}{x-2}\right]\,\mathrm{d}x$$

$$= -\ln(x+1) + \frac{3}{x+1} + \ln(x-2) + C.$$

4.

$$\int \frac{4x^2+x+6}{(x-4)(x^2+4x+5)}\,\mathrm{d}x = \int \left[\frac{2}{x-4} + \frac{2x+1}{x^2+4x+5}\right]\mathrm{d}x$$

$$= 2\ln(x-4) + \ln(x^2+4x+5) - 3\tan^{-1}(x+2) + C.$$

Note:
In the last example above, the second partial fraction has a numerator of $2x+1$ which is not the derivative of x^2+4x+5. But we simply rearrange the numerator as $(2x+4)-3$ to give a third integral which requires the technique of completing the square (discussed in Unit 12.3).

12.6.2 EXERCISES

Integrate the following functions with respect to x:

1. (a)

$$\frac{3x+5}{(x+1)(x+2)};$$

(b)

$$\frac{17x+11}{(x+1)(x-2)(x+3)};$$

(c)

$$\frac{3x^2-8}{(x-1)(x^2+x-7)}.$$

(d)

$$\frac{2x+1}{(x+2)^2(x-3)};$$

(e)

$$\frac{9+11x-x^2}{(x+1)^2(x+2)};$$

(f)

$$\frac{x^5}{(x+2)(x-4)}.$$

2. Evaluate the following definite integrals

(a)

$$\int_2^5 \frac{7x^2+11x+47}{(x-1)(x^2+2x+10)}\,\mathrm{d}x;$$

(b)

$$\int_1^3 \frac{4x^2+1}{x(2x-1)^2}\,\mathrm{d}x.$$

12.6.3 ANSWERS TO EXERCISES

1. (a)

$$2\ln(x+1)+\ln(x+2)+C;$$

(b)

$$\ln(x+1)+3\ln(x-2)-4\ln(x+3)+C;$$

(c)

$$\ln(x-1)+\ln(x^2+x-7)+C;$$

(d)

$$-\frac{3}{5(x+2)}-\frac{7}{25}\ln(x+2)+\frac{7}{25}\ln(x-3)+C;$$

(e)

$$-\frac{3}{(x+1)^2}+\frac{16}{x+1}-\frac{17}{x+2}$$

$$\frac{3}{x+1}+16\ln(x+1)-17\ln(x+2)+C;$$

(f)

$$\frac{x^4}{4}+\frac{2x^3}{3}+6x^2+40x+\frac{16}{3}\ln(x+2)+\frac{512}{3}\ln(x-4)+C.$$

2. (a)

$$\left[5\ln(x-1)+\ln(x^2+2x+10)+\frac{1}{3}\tan^{-1}\frac{x+1}{3}\right]_2^5\simeq -2.726;$$

(b)

$$\left[\ln x-\frac{2}{2x-1}\right]_1^3\simeq 2.699$$