"JUST THE MATHS"

UNIT NUMBER

12.5

INTEGRATION 5
(Integration by parts)

by

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UNIT 12.5 - INTEGRATION 5

INTEGRATION BY PARTS

12.5.1 THE STANDARD FORMULA

The technique to be discussed here provides a convenient method for integrating the product of two functions. However, it is possible to develop a suitable formula by considering, instead, the **derivative** of the product of two functions.

We consider, first, the following comparison:

$\frac{\mathrm{d}}{\mathrm{d}x}[x\sin x] = x\cos x + \sin x$	$\frac{\mathrm{d}}{\mathrm{d}x}[uv] = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$
$x\cos x = \frac{\mathrm{d}}{\mathrm{d}x}[x\sin x] - \sin x$	$u\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}[uv] - v\frac{\mathrm{d}u}{\mathrm{d}x}$
$\int\ x\cos x\ \mathrm{d}x = x\sin x - \int\ \sin x\ \mathrm{d}x$	$\int\ u\frac{\mathrm{d}v}{\mathrm{d}x}\ \mathrm{d}x = uv - \int\ v\frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x$
$= x\sin x + \cos x + C$	

We see that, by labelling the product of two given functions as $u\frac{\mathrm{d}v}{\mathrm{d}x}$, we may express the integral of this product in terms of another integral which, it is anticipated, will be simpler than the original.

To summarise, the formula for **"integration by parts"** is

$$\int\ u\frac{\mathrm{d}v}{\mathrm{d}x}\ \mathrm{d}x = uv - \int\ v\frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x.$$

EXAMPLES

1. Determine

$$I = \int\ x^2e^{3x}\ \mathrm{d}x.$$

Solution

In theory, it does not matter which element of the product x^2e^{3x} is labelled as u and which is labelled as $\frac{\mathrm{d}v}{\mathrm{d}x}$; but the integral obtained on the right-hand-side of the integration by parts formula must be simpler than the original.

In this case we shall take

$$u = x^2\ \text{ and }\ \frac{\mathrm{d}v}{\mathrm{d}x} = e^{3x}.$$

Hence,

$$I = x^2 \frac{e^{3x}}{3} - \int \frac{e^{3x}}{3} .2x \ \mathrm{d}x.$$

That is,

$$I = \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \int xe^{3x} \ \mathrm{d}x.$$

The integral on the right-hand-side still contains the product of two functions and so we must use integration by parts a second time, setting

$$u = x \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = e^{3x}.$$

Thus,

$$I = \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \left[x \frac{e^{3x}}{3} - \int \frac{e^{3x}}{3} .1 \ \mathrm{d}x \right].$$

The integration may now be completed to obtain

$$I = \frac{1}{3} x^2 e^{3x} - \frac{2}{9} xe^{3x} + \frac{2}{27} e^{3x} + C,$$

or

$$I = \frac{e^{3x}}{27} \left[9x^2 - 6x + 2 \right] + C.$$

2. Determine

$$I = \int x \ln x \ \mathrm{d}x.$$

Solution

In this case, we cannot effectively choose $\frac{\mathrm{d}v}{\mathrm{d}x} = \ln x$ since we would need to know the integral of $\ln x$ in order to find v. Hence, we choose

$$u = \ln x \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = x,$$

obtaining

$$I = (\ln x)\frac{x^2}{2} - \int \frac{x^2}{2}.\frac{1}{x}\ \mathrm{d}x.$$

That is,

$$I = \frac{1}{2}x^2 \ln x - \int \frac{x}{2}\ \mathrm{d}x,$$

giving

$$I = \frac{1}{2}x^2 \ln x - \frac{x^2}{4} + C.$$

3. Determine

$$I = \int \ln x\ \mathrm{d}x.$$

Solution
It is possible to regard this as an integration by parts problem if we set

$$u = \ln x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = 1.$$

We obtain

$$I = x \ln x - \int x.\frac{1}{x}\ \mathrm{d}x,$$

giving

$$I = x \ln x - x + C.$$

4. Evaluate

$$I = \int_0^1 \sin^{-1}x\ \mathrm{d}x.$$

Solution
In a similar way to the previous example, it is possible to regard this as an integration by parts problem if we set

$$u = \sin^{-1}x \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = 1.$$

We obtain

$$I = [x\sin^{-1}x]_0^1 - \int_0^1 x.\frac{1}{\sqrt{1-x^2}}\ \mathrm{d}x.$$

That is,

$$I = [x\sin^{-1}x + \sqrt{1-x^2}]_0^1 = \frac{\pi}{2} - 1.$$

5. Determine

$$I = \int e^{2x}\cos x\ \mathrm{d}x.$$

Solution

In this example, it makes little difference whether we choose e^{2x} or $\cos x$ to be u; but we shall set

$$u = e^{2x} \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x.$$

Hence,

$$I = e^{2x}\sin x - \int (\sin x).2e^{2x}\ \mathrm{d}x.$$

That is,

$$I = e^{2x}\sin x - 2\int e^{2x}\sin x\ \mathrm{d}x.$$

Now we need to integrate by parts again, setting

$$u = e^{2x} \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x.$$

Therefore,

$$I = e^{2x}\sin x - 2\left[-e^{2x}\cos x - \int (-\cos x).2e^{2x}\ \mathrm{d}x\right].$$

In other words, the original integral has appeared again on the right hand side to give

$$I = e^{2x}\sin x - 2\left[-e^{2x}\cos x + 2I\right].$$

On simplification,

$$5I = e^{2x}\sin x + 2e^{2x}\cos x,$$

so that

$$I = \frac{1}{5}e^{2x}[\sin x + 2\cos x] + C.$$

Note:

The above examples suggest a priority order for choosing u in a typical integration by parts problem. For example, if the product to be integrated contains a logarithm or an inverse function, then we must choose the logarithm or the inverse function as u; but if there are powers of x without logarithms or inverse functions, then we choose the power of x to be u.

The order of priorities is as follows:

1. LOGARITHMS or INVERSE FUNCTIONS;
2. POWERS OF x;
3. POWERS OF e.

12.5.2 EXERCISES

1. Use integration by parts to evaluate the definite integral

$$\int_0^1 x^3 e^{2x}\ \mathrm{d}x.$$

2. Use integration by parts to integrate the following functions with respect to x:

   (a)

$$x^2\cos 2x;$$

   (b)

$$x^5\ln x;$$

   (c)

$$\tan^{-1}x;$$

(d)

$$x\tan^{-1}x.$$

3. Use integration by parts to evaluate the definite integral

$$\int_0^{\pi} e^{-2x}\sin 3x \ \mathrm{d}x.$$

12.5.3 ANSWERS TO EXERCISES

1.

$$\left[\frac{e^{2x}}{8}(4x^3-6x^2+6x-3)\right]_0^1 = \frac{1}{8}\left(e^2+3\right) \simeq 1.299$$

2. (a)

$$\frac{1}{4}[2x^2\sin 2x + 2x\cos 2x - \sin 2x] + C;$$

(b)

$$\frac{x^6}{36}[6\ln x - 1] + C;$$

(c)

$$x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C;$$

(d)

$$\frac{1}{2}[x^2\tan^{-1}x - x + \tan^{-1}x] + C.$$

3.

$$\left[\frac{e^{-2x}}{13}(3\cos 3x - 2\sin 3x)\right]_0^{\pi} = -\frac{3}{13}(e^{-2\pi}+1) \simeq -0.231$$