"JUST THE MATHS"

UNIT NUMBER

12.3

INTEGRATION 3
(The method of completing the square)

by

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UNIT 12.3 - INTEGRATION 3
THE METHOD OF COMPLETING THE SQUARE

12.3.1 INTRODUCTION AND EXAMPLES

A substitution such as $u = \alpha x + \beta$ may also be used with integrals of the form

$$\int \frac{1}{px^2 + qx + r} \mathrm{d}x \quad \text{and} \quad \int \frac{1}{\sqrt{px^2 + qx + r}} \mathrm{d}x,$$

where, in the first of these, we assume that the quadratic will not factorise into simple linear factors; otherwise the method of partial fractions would be used to integrate it (see Unit 12.6).

Note:
The two types of integral here are often written, for convenience, as

$$\int \frac{\mathrm{d}x}{px^2 + qx + r} \quad \text{and} \quad \int \frac{\mathrm{d}x}{\sqrt{px^2 + qx + r}}.$$

In order to deal with such functions, we shall need to quote standard results which may be deduced from previous ones developed in the differentiation of inverse trigonometric and hyperbolic functions.

They are as follows:

1.
$$\int \frac{1}{a^2 + x^2} \, \mathrm{d}x = \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$$

2.
$$\int \frac{1}{\sqrt{a^2 - x^2}} \, \mathrm{d}x = \sin^{-1} \frac{x}{a} + C.$$

3.
$$\int \frac{1}{\sqrt{a^2 + x^2}} \, \mathrm{d}x = \sinh^{-1} \frac{x}{a} + C \quad \text{or} \quad \ln(x + \sqrt{x^2 + a^2}) + C.$$

4.
$$\int \frac{1}{\sqrt{x^2 - a^2}} \, \mathrm{d}x = \cosh^{-1} \frac{x}{a} + C \quad \text{or} \quad \ln(x + \sqrt{x^2 - a^2}) + C.$$

5.

$$\int \frac{1}{a^2 - x^2}\ \mathrm{d}x = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C;$$

or

$$\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C \quad \text{when} \quad |x| < a,$$

and

$$\frac{1}{2a}\ln\left(\frac{x+a}{x-a}\right) + C \quad \text{when} \quad |x| > a.$$

EXAMPLES

1. Determine the indefinite integral

$$z = \int \frac{\mathrm{d}x}{\sqrt{x^2 + 2x - 3}}.$$

Solution

Completing the square in the quadratic expression gives

$$x^2 + 2x - 3 \equiv (x+1)^2 - 4 \equiv (x+1)^2 - 2^2.$$

Hence,

$$z = \int \frac{\mathrm{d}x}{\sqrt{(x+1)^2 - 2^2}}.$$

Putting $u = x + 1$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$; and so $\frac{\mathrm{d}x}{\mathrm{d}u} = 1$.

Thus,

$$z = \int \frac{\mathrm{d}u}{\sqrt{u^2 - 2^2}},$$

giving

$$z = \ln\left[u + \sqrt{u^2 - 2^2}\right] + C.$$

Returning to the variable, x, we have

$$z = \ln\left[x + 1 + \sqrt{x^2 + 2x - 3}\right] + C.$$

2. Evaluate the definite integral

$$z = \int_3^7 \frac{\mathrm{d}x}{x^2 - 6x + 25}.$$

Solution

Completing the square in the quadratic expression gives

$$x^2 - 6x + 25 \equiv (x-3)^2 + 16.$$

Hence,

$$z = \int_3^7 \frac{\mathrm{d}x}{(x-3)^2 + 16}.$$

Putting $u = x - 3$, we obtain $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$; and so $\frac{\mathrm{d}x}{\mathrm{d}u} = 1$.

Thus,

$$z = \int_0^4 \frac{\mathrm{d}u}{u^2 + 16},$$

giving

$$z = \left[\frac{1}{4}\tan^{-1}\frac{u}{4}\right]_0^4 = \frac{\pi}{16}.$$

Alternatively, without changing the original limits of integration,

$$z = \left[\frac{1}{4}\tan^{-1}\frac{x-3}{4}\right]_3^7.$$

Note:

In cases like the two examples discussed above, when $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and therefore $\frac{\mathrm{d}x}{\mathrm{d}u} = 1$, it seems pointless to go through the laborious process of actually **making** the substitution in detail. All we need to do is to treat the linear expression within the completed square as if it were a single x, then write the result straight down !

12.3.2 EXERCISES

1. Use a table of standard integrals to write down the indefinite integrals of the following functions:

   (a)

   $$\frac{1}{\sqrt{4 - x^2}};$$

   (b)

   $$\frac{1}{9 + x^2};$$

(c)

$$\frac{1}{\sqrt{x^2-7}}.$$

2. By completing the square, evaluate the following definite integrals:

(a)

$$\int_{-1}^{\sqrt{3}-1} \frac{\mathrm{d}x}{x^2+2x+2};$$

(b)

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{3-2x-x^2}}.$$

12.3.3 ANSWERS TO EXERCISES

1. (a)

$$\sin^{-1}\frac{x}{2}+C;$$

(b)

$$\frac{1}{3}\tan^{-1}\frac{x}{3}+C;$$

(c)

$$\ln(x+\sqrt{x^2-7})+C.$$

2. (a)

$$\left[\tan^{-1}(x+1)\right]_{-1}^{\sqrt{3}-1}=\frac{\pi}{3};$$

(b)

$$\left[\sin^{-1}\frac{x+1}{2}\right]_0^1=\frac{\pi}{2}-\frac{\pi}{6}=\frac{\pi}{3}.$$