

“JUST THE MATHS”

UNIT NUMBER

12.2

INTEGRATION 2

(Introduction to definite integrals)

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12.2.1 Definition and examples

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UNIT 12.2 - INTEGRATION 2

INTRODUCTION TO DEFINITE INTEGRALS

12.2.1 DEFINITION AND EXAMPLES

So far, all the integrals considered have been “**indefinite integrals**” since each result has contained an arbitrary constant which cannot be assigned a value without further information.

In practical applications of integration, however, a different kind of integral, called a “**definite integral**”, is encountered and is represented by a numerical value rather than a function plus an arbitrary constant.

Suppose that

$$\int f(x)dx = g(x) + C.$$

Then the symbol

$$\int_a^b f(x)dx$$

is used to mean

(Value of $g(x) + C$ at $x = b$) minus (Value of $g(x) + C$ at $x = a$).

In other words, since C will cancel out,

$$\int_a^b f(x)dx = g(b) - g(a).$$

The right hand side of this statement can also be written

$$[g(x)]_a^b,$$

a notation which is used as the middle stage of a definite integral calculation.

The values a and b are known as the “**lower limit**” and “**upper limit**”, respectively, of the definite integral (even when a is larger than b).

EXAMPLES

1. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos x dx.$$

Solution

$$\int_0^{\frac{\pi}{2}} \cos x dx = [\sin x]_0^{\frac{\pi}{2}} = \sin \frac{\pi}{2} - \sin 0 = 1.$$

2. Evaluate the definite integral

$$\int_1^3 (2x + 1)^2 dx.$$

Solution

Using the method of integrating a function of a linear function, we obtain

$$\int_1^3 (2x + 1)^2 dx = \left[\frac{(2x + 1)^3}{6} \right]_1^3 = \frac{7^3}{6} - \frac{3^3}{6} \simeq 52.67$$

Notes:

(i) If we had decided to multiply out the integrand $(2x + 1)^2$ before integrating, giving

$$4x^2 + 4x + 1,$$

the integration process would have yielded the expression

$$4\frac{x^3}{3} + 2x^2 + x,$$

which differs only from the previous result by the constant value $\frac{1}{6}$; students may like to check this. Hence the numerical result for the definite integral will be the same.

(ii) An alternative method of evaluating the definite integral would be to make the substitution

$$u = 2x + 1.$$

But, whenever substitution is used for definite integrals, it is not necessary to return to the original variable at the end as long as the limits of integration are changed to the appropriate values for u .

Replacing dx by $\frac{dx}{du} du$ (that is, $\frac{1}{2} du$) and the limits $x = 1$ and $x = 3$ by $u = 2 \times 1 + 1 = 3$ and $u = 2 \times 3 + 1 = 7$, respectively, we obtain

$$\int_3^7 u^2 \frac{1}{2} du = \left[\frac{u^3}{6} \right]_3^7 = \frac{7^3}{6} - \frac{3^3}{6} \simeq 52.67$$

12.2.2 EXERCISES

Evaluate the following definite integrals:

1.

$$\int_0^{\frac{\pi}{3}} \sin 4x dx.$$

2.

$$\int_{-1}^1 (x + 1)^7 dx.$$

3.

$$\int_0^{\frac{\pi}{4}} \sec^2(x + \pi) dx.$$

4.

$$\int_1^2 \frac{1}{4 + 3x} dx.$$

5.

$$\int_0^1 e^{1-7x} dx.$$

12.2.3 ANSWERS TO EXERCISES

1.

0.375

2.

32.

3.

1.

4.

$\frac{1}{3} \ln \frac{10}{7}$ or 0.119 approximately

5.

0.388 approximately