"JUST THE MATHS"

UNIT NUMBER

12.10

INTEGRATION 10
(Further reduction formulae)

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UNIT 12.10 - INTEGRATION 10

FURTHER REDUCTION FORMULAE

INTRODUCTION

As an extension to the idea of reduction formulae, there are two particular definite integrals which are worthy of special consideration. They are

$$\int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^n x \, \mathrm{d}x.$$

But, first, we shall establish the reduction formulae for the equivalent indefinite integrals.

12.10.1 INTEGER POWERS OF A SINE

Suppose that

$$I_n = \int \sin^n x \, \mathrm{d}x;$$

then, by writing the integrand as the product of two functions, we have

$$I_n = \int \sin^{n-1} x \sin x \, \mathrm{d}x.$$

Using integration by parts, with $u = \sin^{n-1} x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$, we obtain

$$I_n = \sin^{n-1} x(-\cos x) + \int (n-1)\sin^{n-2} x \cos^2 x \, \mathrm{d}x.$$

But, since $\cos^2 x \equiv 1 - \sin^2 x$, this becomes

$$I_n = -\sin^{n-1} x \cos x + (n-1)[I_{n-2} - I_n].$$

Thus,

$$I_n = \frac{1}{n}\left[-\sin^{n-1} x \cos x + (n-1)I_{n-2}\right].$$

EXAMPLE

Determine the indefinite integral

$$\int \sin^6 x \, \mathrm{d}x.$$

Solution

$$I_6 = \frac{1}{6}\left[-\sin^5 x \cos x + 5I_4\right],$$

where

$$I_4 = \frac{1}{4}\left[-\sin^3 x \cos x + 3I_2\right], \quad I_2 = \frac{1}{2}\left[-\sin x \cos x + I_0\right]$$

and

$$I_0 = \int \mathrm{d}x = x + \text{constant}.$$

Hence,

$$I_2 = \frac{1}{2}\left[-\sin x \cos x + x + \text{constant}\right];$$

$$I_4 = \frac{1}{4}\left[-\sin^3 x \cos x - \frac{3}{2}\sin x \cos x + \frac{3}{2}x + \text{constant}\right];$$

$$I_6 = \frac{1}{6}\left[-\sin^5 x \cos x - \frac{5}{4}\sin^3 x \cos x - \frac{15}{8}\sin x \cos x + \frac{15}{8}x + \text{constant}\right].$$

$$\text{Thus,} \quad \int \sin^6 x \, \mathrm{d}x = -\frac{1}{6}\sin^5 x \cos x - \frac{5}{24}\sin^3 x \cos x - \frac{5}{16}\sin x \cos x + \frac{5x}{16} + C,$$

where C is an arbitrary constant.

12.10.2 INTEGER POWERS OF A COSINE

Suppose that

$$I_n = \int \cos^n x \; \mathrm{d}x;$$

then, by writing the integrand as the product of two functions, we have

$$I_n = \int \cos^{n-1} x \cos x \; \mathrm{d}x.$$

Using integration by parts, with $u = \cos^{n-1} x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$, we obtain

$$I_n = \cos^{n-1} x \sin x + \int (n-1) \cos^{n-2} x \sin^2 x \; \mathrm{d}x.$$

But, since $\sin^2 x \equiv 1 - \cos^2 x$, this becomes

$$I_n = \cos^{n-1} x \sin x + (n-1)[I_{n-2} - I_n].$$

Thus,

$$I_n = \frac{1}{n} \left[\cos^{n-1} x \sin x + (n-1) I_{n-2} \right].$$

EXAMPLE

Determine the indefinite integral

$$\int \cos^5 x \; \mathrm{d}x.$$

Solution

$$I_5 = \frac{1}{5} \left[\cos^4 x \sin x + 4 I_3 \right],$$

where

$$I_3 = \frac{1}{3}\left[\cos^2 x \sin x + 2I_1\right]$$

and

$$I_1 = \int \cos x \, \mathrm{d}x = \sin x + \text{constant}.$$

Hence,

$$I_3 = \frac{1}{3}\left[\cos^2 x \sin x + 2\sin x + \text{constant}\right];$$

$$I_5 = \frac{1}{5}\left[\cos^4 x \sin x + \frac{4}{3}\cos^2 x \sin x + \frac{8}{3}\sin x + \text{constant}\right];$$

We conclude that

$$\int \cos^5 x \, \mathrm{d}x = \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\sin x + C,$$

where C is an arbitrary constant.

12.10.3 WALLIS'S FORMULAE

Here, we consider the definite integrals

$$\int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^n x \, \mathrm{d}x.$$

Denoting either of these integrals by I_n, the reduction formula reduces to

$$I_n = \frac{n-1}{n} I_{n-2}$$

in both cases, from the previous two sections.

Convenient results may be obtained from this formula according as n is an odd number or an even number, as follows:

(a) n is an odd number

Repeated application of the reduction formula gives

$$I_n = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}. \ldots \frac{6}{7}.\frac{4}{5}.\frac{2}{3}.I_1.$$

But

$$I_1 = \int_0^{\frac{\pi}{2}} \sin x \, \mathrm{d}x \quad \text{or} \quad I_1 = \int_0^{\frac{\pi}{2}} \cos x \, \mathrm{d}x,$$

both of which have a value of 1.

Therefore,

$$I_n = \frac{(n-1)(n-3)(n-5). \ldots 6.4.2}{n(n-2)(n-4). \ldots 7.5.3},$$

which is **the first of "Wallis's formulae"**.

(b) n is an even number

This time, repeated application of the reduction formula gives

$$I_n = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}. \ldots \frac{5}{6}.\frac{3}{4}.\frac{1}{2}.I_0.$$

But

$$I_0 = \int_0^{\frac{\pi}{2}} \mathrm{d}x = \frac{\pi}{2}.$$

Therefore,

$$I_n = \frac{(n-1)(n-3)(n-5). \ldots 5.3.1}{n(n-2)(n-4). \ldots 6.4.2} \frac{\pi}{2},$$

which is **the second of “Wallis’s formulae”**.

EXAMPLES

1. Evaluate the definite integral
$$\int_0^{\frac{\pi}{2}} \sin^5 x \, \mathrm{d}x.$$

   Solution
$$\int_0^{\frac{\pi}{2}} \sin^5 x \, \mathrm{d}x = \frac{4.2}{5.3} = \frac{8}{15}.$$

2. Evaluate the definite integral
$$\int_0^{\frac{\pi}{2}} \cos^4 x \, \mathrm{d}x.$$

   Solution
$$\int_0^{\frac{\pi}{2}} \cos^4 x \, \mathrm{d}x = \frac{3.1}{4.2}\frac{\pi}{2} = \frac{3\pi}{16}.$$

12.10.4 COMBINATIONS OF SINES AND COSINES

Another type of problem to which Wallis’s formulae may be applied is of the form

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, \mathrm{d}x,$$

where either m or n (or both) is an even number. We simply use $\sin^2 x \equiv 1 - \cos^2 x$ or $\cos^2 x \equiv 1 - \sin^2 x$ in order to convert the problem to several integrals of the types already discussed.

EXAMPLE

Evaluate the definite integral
$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin^2 x \, \mathrm{d}x.$$

Solution

$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin^2 x \, \mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^5 x \left(1 - \cos^2 x\right) \, \mathrm{d}x = \int_0^{\frac{\pi}{2}} \left(\cos^5 x - \cos^7 x\right) \, \mathrm{d}x,$$

which may be interpreted as

$$I_5 - I_7 = \frac{4.2}{5.3} - \frac{5.4.3}{6.4.2} = \frac{8}{15} - \frac{16}{35} = \frac{8}{105}.$$

12.10.5 EXERCISES

1. Determine the indefinite integral

$$\int \cos^4 x \, \mathrm{d}x.$$

2. Determine the indefinite integral

$$\int \sin^7 x \, \mathrm{d}x.$$

3. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \sin^6 x \, \mathrm{d}x.$$

4. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^9 x \, \mathrm{d}x.$$

5. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^2 x \sin^6 x \, \mathrm{d}x.$$

6. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cos^4 x \, \mathrm{d}x.$$

12.10.6 ANSWERS TO EXERCISES

1.

$$\frac{1}{4}\cos^3 x\sin x + \frac{3}{8}\cos x\sin x + \frac{3x}{8} + C.$$

2.

$$-\frac{1}{7}\sin^6 x\cos x - \frac{6}{35}\sin^4 x\cos x - \frac{24}{105}\sin^2 x\cos x - \frac{16}{35}\cos x + C.$$

3.

$$\frac{5\pi}{32}.$$

4.

$$\frac{128}{315}.$$

5.

$$\frac{5\pi}{32}.$$

6.

$$-\frac{4}{105}.$$