"JUST THE MATHS"

UNIT NUMBER

11.5

DIFFERENTIATION APPLICATIONS 5
(Maclaurin's and Taylor's series)

by

A.J.Hobson

UNIT 11.5 - DIFFERENTIATION APPLICATIONS 5

MACLAURIN'S AND TAYLOR'S SERIES

11.5.1 MACLAURIN'S SERIES

One of the simplest kinds of function to deal with, in either algebra or calculus, is a polynomial (see Unit 1.8). Polynomials are easy to substitute numerical values into and they are easy to differentiate.

One useful application of the present section is to approximate, to a polynomial, functions which are not already in polynomial form.

THE GENERAL THEORY

Suppose $f(x)$ is a given function of x which is not in the form of a polynomial, and let us assume that it may be expressed in the form of an infinite series of ascending powers of x; that is, a **"power series"**, (see Unit 2.4).

More specifically, we assume that

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ \dots$$

This assumption cannot be justified unless there is a way of determining the **"coefficients"**, a_0, a_1, a_2, a_3, a_4, etc.; but this is possible as an application of differentiation as we now show:

(a) Firstly, if we substitute $x = 0$ into the assumed formula for $f(x)$, we obtain $f(0) = a_0$; in other words,

$$a_0 = f(0).$$

(b) Secondly, if we differentiate the assumed formula for $f(x)$ once with respect to x, we obtain

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots$$

which, on substituting $x = 0$, gives $f'(0) = a_1$; in other words,

$$a_1 = f'(0).$$

(c) Differentiating a second time leads to the result that

$$f''(x) = 2a_2 + (3 \times 2)a_3 x + (4 \times 3)a_4 x^2 + \ldots$$

which, on substituting $x = 0$ gives $f''(0) = 2a_2$; in other words,

$$a_2 = \frac{1}{2} f''(0).$$

(d) Differentiating yet again leads to the result that

$$f'''(x) = (3 \times 2)a_3 + (4 \times 3 \times 2)a_4 x + \ldots$$

which, on substituting $x = 0$ gives $f'''(0) = (3 \times 2)a_3$; in other words,

$$a_3 = \frac{1}{3!} f'''(0).$$

(e) Continuing this process with further differentiation will lead to the general formula

$$a_n = \frac{1}{n!} f^{(n)}(0),$$

where $f^{(n)}(0)$ means the value, at $x = 0$ of the n-th derivative of $f(x)$.

Summary

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

This is called the **"Maclaurin's series for** $f(x)$**"**.

Notes:

(i) We must assume, ofcourse, that all of the derivatives of $f(x)$ exist at $x = 0$ in the first place; otherwise the above result is invalid.
It is also necessary to examine, for convergence or divergence, the Maclaurin's series obtained

for a particular function. The result may not be used when the series diverges; (see Units 2.3 and 2.4).

(b) If x is small and it is possible to neglect powers of x after the n-th power, then Maclaurin's series approximates $f(x)$ to a polynomial of degree n.

11.5.2 STANDARD SERIES

Here, we determine the Maclaurin's series for some of the functions which occur frequently in the applications of mathematics to science and engineering. The ranges of values of x for which the results are valid will be stated without proof.

1. **The Exponential Series**

   (i) $f(x) \equiv e^x$; hence, $f(0) = e^0 = 1$.

   (ii) $f'(x) = e^x$; hence, $f'(0) = e^0 = 1$.

   (iii) $f''(x) = e^x$; hence, $f''(0) = e^0 = 1$.

   (iv) $f'''(x) = e^x$; hence, $f'''(0) = e^0 = 1$.

   (v) $f^{(iv)}(x) = e^x$; hence, $f^{(iv)}(0) = e^0 = 1$.

   Thus,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ \ldots$$

   and it may be shown that this series is valid for all values of x.

2. **The Sine Series**

   (i) $f(x) \equiv \sin x$; hence, $f(0) = \sin 0 = 0$.

   (ii) $f'(x) = \cos x$; hence, $f'(0) = \cos 0 = 1$.

   (iii) $f''(x) = -\sin x$; hence, $f''(0) = -\sin 0 = 0$.

   (iv) $f'''(x) = -\cos x$; hence, $f'''(0) = -\cos 0 = -1$.

   (v) $f^{(iv)}(x) = \sin x$; hence, $f^{(iv)}(0) = \sin 0 = 0$.

   (vi) $f^{(v)}(x) = \cos x$; hence, $f^{(v)}(0) = \cos 0 = 1$.

   Thus,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ \ldots$$

   and it may be shown that this series is valid for all values of x.

3. The Cosine Series

(i) $f(x) \equiv \cos x$; hence, $f(0) = \cos 0 = 1$.

(ii) $f'(x) = -\sin x$; hence, $f'(0) = -\sin 0 = 0$.

(iii) $f''(x) = -\cos x$; hence, $f''(0) = -\cos 0 = -1$.

(iv) $f'''(x) = \sin x$; hence, $f'''(0) = \sin 0 = 0$.

(v) $f^{(iv)}(x) = \cos x$; hence, $f^{(iv)}(0) = \cos 0 = 1$.

Thus,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ \ldots$$

and it may be shown that this series is valid for all values of x.

4. The Logarithmic Series

It is not possible to find a Maclaurin's series for the function $\ln x$, since neither the function nor its derivatives exist at $x = 0$.

As an alternative, we may consider the function $\ln(1 + x)$ instead.

(i) $f(x) \equiv \ln(1 + x)$; hence, $f(0) = \ln 1 = 0$.

(ii) $f'(x) = \frac{1}{1+x}$; hence, $f'(0) = 1$.

(iii) $f''(x) = -\frac{1}{(1+x)^2}$; hence, $f''(0) = 1$.

(iv) $f'''(x) = \frac{2}{(1+x)^3}$; hence, $f'''(0) = 2$.

(v) $f^{(iv)}(x) = -\frac{2\times 3}{(1+x)^4}$; hence, $f^{(iv)}(0) = -(2 \times 3)$.

Thus,

$$\ln(1 + x) = x - \frac{x^2}{2!} + 2\frac{x^3}{3!} - (2 \times 3)\frac{x^4}{4!} + \ \ldots$$

which simplifies to

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ \ldots$$

and it may be shown that this series is valid for $-1 < x \leq 1$.

5. The Binomial Series

The statement of the Binomial Formula has already appeared in Unit 2.2; and it was seen there that

(a) When n is a positive integer, the expansion of $(1 + x)^n$ in ascending powers of x is a **finite** series;

(b) When n is a negative integer or a fraction, the expansion of $(1+x)^n$ in ascending powers of x is an **infinite** series.

Here, we examine the proof of the Binomial Formula.

(i) $f(x) \equiv (1+x)^n$; hence, $f(0) = 1$.

(ii) $f'(x) = n(1+x)^{n-1}$; hence, $f'(0) = n$.

(iii) $f''(x) = n(n-1)(1+x)^{n-2}$; hence, $f''(0) = n(n-1)$.

(iv) $f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$; hence, $f'''(0) = n(n-1)(n-2)$.

(v) $f^{(iv)}(x) = n(n-1)(n-2)(n-3)(1+x)^{n-4}$; hence, $f^{(iv)}(0) = n(n-1)(n-2)(n-3)$.

Thus,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ \ldots$$

If n is a positive integer, all of the derivatives of $(1+x)^n$ after the n-th derivative are identically equal to zero; so the series is a finite series ending with the term in x^n.

In all other cases, the series is an infinite series and it may be shown that it is valid whenever $-1 < x \leq 1$.

EXAMPLES

1. Use the Maclaurin's series for $\sin x$ to evaluate

$$\lim_{x\to 0} \frac{x + \sin x}{x(x+1)}.$$

Solution

Substituting the series for $\sin x$ gives

$$\lim_{x\to 0} \frac{x + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ \ldots}{x^2 + x}$$

$$= \lim_{x\to 0} \frac{2x - \frac{x^3}{6} + \frac{x^5}{120} - \ \ldots}{x^2 + x}$$

$$= \lim_{x\to 0} \frac{2 - \frac{x^2}{6} + \frac{x^4}{120} - \ \ldots}{x + 1} = 2.$$

2. Use a Maclaurin's series to evaluate $\sqrt{1.01}$ correct to six places of decimals.

   Solution

   We shall consider the expansion of the function $(1+x)^{\frac{1}{2}}$ and then substitute $x = 0.01$.

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \ \ldots$$

   That is,

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ \ldots$$

   Substituting $x = 0.01$ gives

$$\sqrt{1.01} = 1 + \frac{1}{2} \times 0.01 - \frac{1}{8} \times 0.0001 + \frac{1}{16} \times 0.000001 - \ \ldots$$

$$= 1 + 0.005 - 0.0000125 + 0.0000000625 - \ \ldots$$

   The fourth term will not affect the sixth decimal place in the result given by the first three terms; and this is equal to 1.004988 correct to six places of decimals.

3. Assuming the Maclaurin's series for e^x and $\sin x$ and assuming that they may be multiplied together term-by-term, obtain the expansion of $e^x \sin x$ in ascending powers of x as far as the term in x^5.

   Solution

$$e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{120} + \ldots\right)$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120} + x^2 - \frac{x^4}{6} + \frac{x^3}{2} - \frac{x^5}{12} + \frac{x^4}{6} + \frac{x^5}{24} + \ \ldots$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \ \ldots$$

11.5.3 TAYLOR'S SERIES

A useful consequence of Maclaurin's series is known as Taylor's series and one form of it may be stated as follows:

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \ . \ . \ .$$

Proof:

To obtain this result from Maclaurin's series, we simply let $f(x+h) \equiv F(x)$. Then,

$$F(x) = F(0) + xF'(0) + \frac{x^2}{2!}F''(0) + \frac{x^3}{3!}F'''(0) + \ . \ .$$

But, $F(0) = f(h)$, $F'(0) = f'(h)$, $F''(0) = f''(h)$, $F'''(0) = f'''(h)$,. . . which proves the result.

Note: An alternative form of Taylor's series, often used for approximations, may be obtained by interchanging the symbols x and h to give

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ . \ . \ .$$

EXAMPLE

Given that $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, use Taylor's series to evaluate $\sin(x+h)$, correct to five places of decimals, in the case when $x = \frac{\pi}{4}$ and $h = 0.01$.

Solution

Using the sequence of derivatives as in the Maclaurin's series for $\sin x$, we have

$$\sin(x+h) = \sin x + h\cos x - \frac{h^2}{2!}\sin x - \frac{h^3}{3!}\cos x + ...$$

Subsituting $x = \frac{\pi}{4}$ and $h = 0.01$, we obtain

$$\sin\left(\frac{\pi}{4} + 0.01\right) = \frac{1}{\sqrt{2}}\left(1 + 0.01 - \frac{(0.01)^2}{2!} - \frac{(0.01)^3}{3!} + \ . \ . \ .\right)$$

$$= \frac{1}{\sqrt{2}}(1 + 0.01 - 0.00005 - 0.000000017 + ...)$$

The fourth term does not affect the fifth decimal place in the sum of the first three terms; and so

$$\sin\left(\frac{\pi}{4}+0.01\right) \simeq \frac{1}{\sqrt{2}} \times 1.00995 \simeq 0.71414$$

11.5.4 EXERCISES

1. Determine the first three non-vanishing terms of the Maclaurin's series for the function $\sec x$.
2. Determine the Maclaurin's series for the function $\tan x$ as far as the term in x^5.
3. Determine the Maclaurin's series for the function $\ln(1+e^x)$ as far as the term in x^4.
4. Use the Maclaurin's series for the function e^x to deduce the expansion, in ascending powers of x of the function e^{-x} and then use these two series to obtain the expansion, in ascending powers of x, of the functions

   (a)

   $$\frac{e^x+e^{-x}}{2}(\equiv \cosh x);$$

   (b)

   $$\frac{e^x-e^{-x}}{2}(\equiv \sinh x).$$

5. Use the Maclaurin's series for the function $\cos x$ and the Binomial Series for the function $\frac{1}{1+x}$ to obtain the expansion of the function

   $$\frac{\cos x}{1+x}$$

   in ascending powers of x as far as the term in x^4.
6. From the Maclaurin's series for the function $\cos x$, deduce the expansions of the functions $\cos 2x$ and $\sin^2 x$ as far as the term in x^4.

7. Use appropriate Maclaurin's series to evaluate the following limits:

   (a)
$$\lim_{x \to 0} \left[\frac{e^x + e^{-x} - 2}{2\cos 2x - 2} \right];$$

   (b)
$$\lim_{x \to 0} \left[\frac{\sin^2 x - x^2 \cos x}{x^4} \right].$$

8. Use a Maclaurin's series to evaluate $\sqrt[3]{1.05}$ correct to four places of decimals.

9. Expand $\cos(x + h)$ as a series of ascending powers of h.

   Given that $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, evaluate $\cos(x + h)$, correct to five places of decimals, in the case when $x = \frac{\pi}{6}$ and $h = -0.05$.

11.5.5 ANSWERS TO EXERCISES

1.
$$1 + \frac{x^2}{2} + \frac{x^4}{8} + \ . \ .$$

2.
$$x + \frac{x^3}{3} + \frac{2x^5}{15} + \ . \ . \ .$$

3.
$$\ln 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \ . \ . \ .$$

4. (a)
$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ . \ . \ .;$$

   (b)
$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ . \ .$$

5.

$$1 - x + \frac{x^2}{2} - \frac{x^3}{2} + \frac{13x^4}{24} - \ \ldots$$

6.

$$\cos 2x = 1 - 2x^2 + \frac{2x^4}{3} - \ \ldots$$

$$\sin^2 x = x^2 - \frac{x^4}{3} + \ \ldots$$

7. (a) $-\frac{1}{4}$, (b) $\frac{1}{6}$

8. 1.0164

9. 0.74156