“JUST THE MATHS”

UNIT NUMBER

11.6

DIFFERENTIATION APPLICATIONS 6
(Small increments and small errors)

by

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UNIT 11.6 - DIFFERENTIATION APPLICATIONS 6

SMALL INCREMENTS AND SMALL ERRORS

11.6.1 SMALL INCREMENTS

Given that a dependent variable, y, and an independent variable, x are related by means of the formula

$$y = f(x),$$

suppose that x is subject to a small **"increment"**, δx,

In the present context we use the term "increment" to mean that δx is positive when x is **increased**, but negative when x is **decreased**.

The exact value of the corresponding increment, δy, in y is given by

$$\delta y = f(x + \delta x) - f(x),$$

but this can often be a cumbersome expression to evaluate.

However, since δx is small, we may recall, from the definition of a derivative (Unit 10.2), that

$$\frac{f(x + \delta x) - f(x)}{\delta x} \simeq \frac{\mathrm{d}y}{\mathrm{d}x}.$$

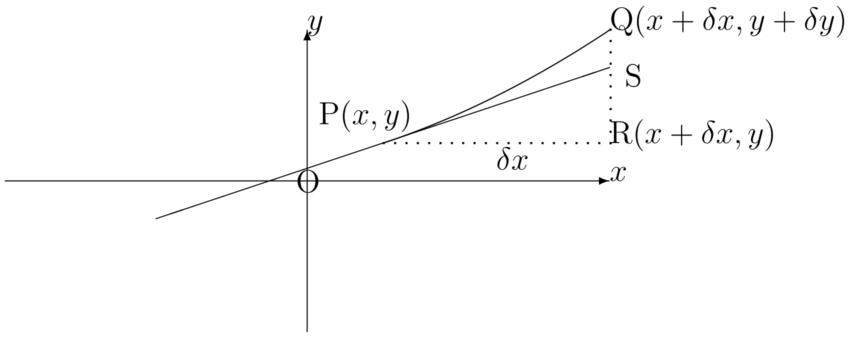
That is,

$$\frac{\delta y}{\delta x} \simeq \frac{\mathrm{d}y}{\mathrm{d}x};$$

and we may conclude that

$$\delta y \simeq \frac{\mathrm{d}y}{\mathrm{d}x}\delta x.$$

For a diagramatic approach to this approximation for the increment in y, let us consider the graph of y against x in the neighbourhood of the two points $\mathrm{P}(x, y)$ and $\mathrm{Q}(x + \delta x, y + \delta y)$ on the curve whose equation is $y = f(x)$.



In the diagram, PR $= \delta x$, QR $= \delta y$ and the gradient of the line PS is given by the value of $\frac{dy}{dx}$ at P.

Taking SR as an approximation to QR, we obtain

$$\frac{\text{SR}}{\text{PR}} = \left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{\text{P}}.$$

In other words,

$$\frac{\text{SR}}{\delta x} = \left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{\text{P}}.$$

Hence,

$$\delta y \simeq \left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{\text{P}} \delta x,$$

which is the same result as before.

Notes:

(i) The quantity $\frac{\mathrm{d}y}{\mathrm{d}x}\delta x$ is known as the **"total differential of y"** (or simply the "differential of y"). It provides an approximation **(including the appropriate sign)** for the increment, δy, in y subject to an increment of δx in x.

(ii) It is important **not** to use the word "differential" when referring to a "derivative". Rather, the correct alternative to "derivative" is "differential coefficient".

(iii) A more rigorous approach to the calculation of δy is to use the result known as "Taylor's Theorem" (see Unit 11.5) which, in this context, would give the formula

$$f(x+\delta x) = f(x) + f'(x)\delta x + \frac{f''(x)}{2!}(\delta x)^2 + \frac{f'''(x)}{3!}(\delta x)^3 + \ldots .$$

Hence, if δx is small enough for powers of two and above to be neglected, then

$$f(x+\delta x) - f(x) \simeq f'(x)\delta x$$

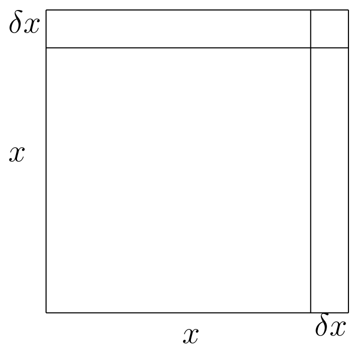
to the first order of approximation.

EXAMPLES

1. If a square has side $x$cms., determine both the exact and the approximate values of the increment in the area Acms^2. when x is increased by δx.

   Solution

   (a) Exact Method



The area is given by the formula

$$A = x^2.$$

If x increases by δx, then the increase, δA, in A may be obtained from the formula

$$A + \delta A = (x + \delta x)^2 = x^2 + 2x\delta x + (\delta x)^2.$$

That is,

$$\delta A = 2x\delta x + (\delta x)^2.$$

(b) Approximate Method

Here, we use

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 2x$$

to give

$$\delta A \simeq 2x\delta x;$$

and we observe from the diagram that the two results differ only by the area of the small square, with side δx.

2. If

$$y = xe^{-x},$$

calculate, approximately, the change in y when x increases from 5 to 5.03.

Solution

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{-x}(1 - x),$$

so that

$$\delta y \simeq e^{-x}(1 - x)\delta x,$$

where $x = 5$ and $\delta x = 0.3$.

Hence,

$$\delta y \simeq e^{-5}.(1 - 5).(0.3) \simeq -0.00809,$$

showing a **decrease** of 0.00809 in y.

We may compare this with the exact value which is given by

$$\delta y = 5.3e^{-5.3} - 5e^{-5} \simeq -0.00723$$

3. If

$$y = xe^{-x},$$

determine, in terms of x, the percentage change in y when x is increased by 2%.

Solution

Once again, we have

$$\delta y = e^{-x}(1 - x)\delta x;$$

but, this time, $\delta x = 0.02x$, so that

$$\delta y = e^{-x}(1 - x) \times 0.02x.$$

The **percentage** change in y is given by

$$\frac{\delta y}{y} \times 100 = \frac{e^{-x}(1 - x) \times 0.02x}{xe^{-x}} \times 100 = 2(1 - x).$$

That is, y increases by $2(1-x)$%, which will be positive when $x < 1$ and negative when $x > 1$.

Note:

It is usually more meaningful to discuss increments in the form of a percentage, since this gives a better idea of how much a variable has changed in porportion to its original value.

11.6.2 SMALL ERRORS

In the functional relationship

$$y = f(x),$$

let us suppose that x is known to be subject to an error in measurement; then we consider what error will be likely in the calculated value of y.

In particular, suppose x is known to be **too large** by a small amount, δx, in which case the correct value of x could be obtained if we **decreased** it by δx; or, what amounts to the same thing, if we **increased** it by $-\delta x$.

Correspondingly, the value of y will **increase** by approximately $-\frac{dy}{dx}\delta x$; that is, y will **decrease** by approximately $\frac{dy}{dx}\delta x$.

Summary

We conclude that, if x is too large by an amount δx, then y is too large by approximately $\frac{\mathrm{d}y}{\mathrm{d}x}\delta x$; though, ofcourse, if $\frac{\mathrm{d}y}{\mathrm{d}x}$ itself is negative, y will be too small when x is too large and vice versa.

EXAMPLES

1. If

$$y = x^2 \sin x,$$

calculate, approximately, the error in y when x is measured as 3, but this measurement is subsequently discovered to be too large by 0.06.

Solution

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 \cos x + 2x \sin x$$

and, hence,

$$\delta y \simeq (x^2 \cos x + 2x \sin x)\delta x,$$

where $x = 3$ and $\delta x = 0.06$.

The error in y is therefore given approximately by

$$\delta y \simeq (3^2 \cos 3 + 6 \sin 3) \times 0.06 \simeq -0.4838$$

That is, y is too small by approximately 0.4838.

2. If

$$y = \frac{x}{1+x},$$

determine approximately, in terms of x, the percentage error in y when x is subject to an error of 5%.

Solution

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+x-x}{(1+x)^2} = \frac{1}{(1+x)^2},$$

so that

$$\delta y \simeq \frac{1}{(1+x)^2}\delta x,$$

where $\delta x = 0.05x$.

The **percentage** error in y is thus given by

$$\frac{\delta y}{y} \times 100 \simeq \frac{1}{(1+x)^2} \times 0.05x \times \frac{x+1}{x} \times 100 = \frac{5}{1+x}.$$

Hence, y is too large by approximately $\frac{5}{1+x}\%$ which will be positive when $x > -1$ and negative when $x < -1$.

11.6.3 EXERCISES

1. If

$$y = \frac{e^{2x}}{x},$$

calculate, approximately, the change in y when x is increased from 1 to 1.0025.

State your answer correct to three significant figures.

2. If

$$y = (2x+1)^5,$$

determine approximately, in terms of x, the percentage change in y when x increases by 0.1%.

3. If

$$y = x^3 \ln x,$$

calculate approximately, correct to the nearest integer, the error in y when x is measured as 4, but this measurement is subsequently discovered to be too small by 0.12.

4. If

$$y = \cos(3x^2 + 2),$$

determine approximately, in terms of x, the percentage error in y if x is too large by 2%.

You may assume that $3x^2 + 2$ lies between π and $\frac{3\pi}{2}$.

11.6.4 ANSWERS TO EXERCISES

1. y increases by approximately 0.0185.
2. y increases by approximately $\frac{x}{(2x+1)}\%$
3. y is too small by approximately 10.
4. y is too small by approximately $-12x^2\tan(3x^2+2)$.