"JUST THE MATHS"

UNIT NUMBER

11.4

DIFFERENTIATION APPLICATIONS 4
(Circle, radius & centre of curvature)

by

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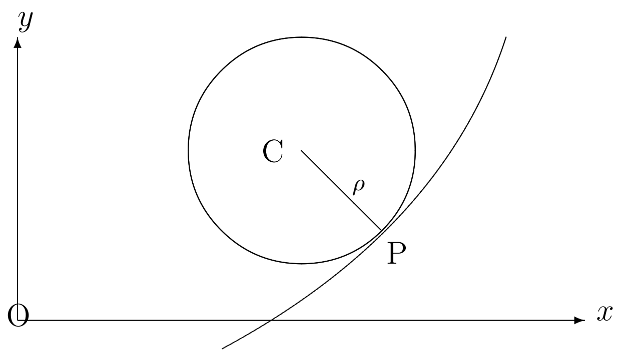
UNIT 11.4 DIFFERENTIATION APPLICATIONS 4

CIRCLE, RADIUS AND CENTRE OF CURVATURE

11.4.1 INTRODUCTION

At a point, P, on a given curve, suppose we were to draw a circle which **just touches** the curve and has the same value of the curvature (including its sign). This circle is called the **"circle of curvature at P"**. Its radius, ρ, is called the **"radius of curvature at P"** and its centre is called the **"centre of curvature at P"**.



11.4.2 RADIUS OF CURVATURE

Using the earlier examples on the circle (Unit 11.3), we conclude that, if the curvature at P is κ, then $\rho = \frac{1}{\kappa}$ and, hence,

$$\rho = \frac{\mathrm{d}s}{\mathrm{d}\theta}.$$

Furthermore, in cartesian co-ordinates,

$$\rho = \frac{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}.$$

Note:
If we are interested in the radius of curvature simply as a length, then, for curves with

negative curvature, we would use only the **numerical** value obtained in the above formula. However, in a later discussion, it is necessary to use the appropriate sign for the radius of curvature.

EXAMPLE

Calculate the radius of curvature at the point $(0.5, -1)$ of the curve whose equation is

$$y^2 = 2x.$$

Solution

Differentiating implicitly,

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2.$$

That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y}.$$

Also

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{y^2}.\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{y^3}.$$

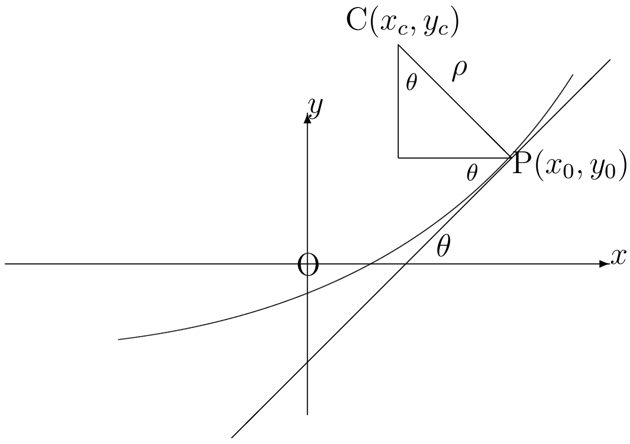
Hence, at the point $(0.5, -1)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1$.

We conclude that

$$\rho = \frac{(1+1)^{\frac{3}{2}}}{1} = 2\sqrt{2}.$$

11.4.3 CENTRE OF CURVATURE

We shall consider a point, (x_0, y_0), on an arc of a curve whose equation is $y = f(x)$ and for which the curvature is positive, the arc lying in the first quadrant. But it may be shown that the formulae obtained for the co-ordinates, (x_c, y_c), of the centre of curvature apply in any situation, provided that the curvature is associated with its appropriate sign.



From the diagram,

$$
\begin{aligned}
x_c &= x_0 - \rho \sin\theta, \\
y_c &= y_0 + \rho \cos\theta.
\end{aligned}
$$

Note:
Although the formulae apply in any situation, it is a good idea to sketch the curve in order estimate, roughly, where the centre of curvature is going to be. This is especially important where there is uncertainty about the precise value of the angle θ.

EXAMPLE

Determine the centre of curvature at the point $(0.5, -1)$ of the curve whose equation is

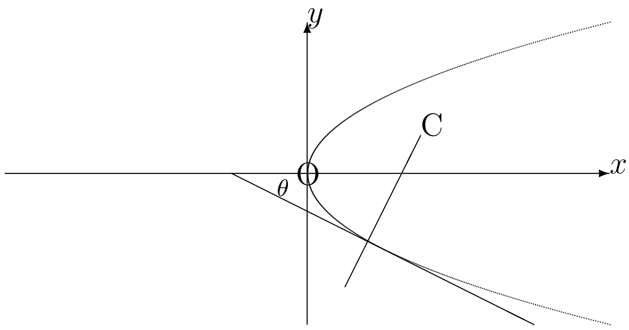
$$y^2 = 2x.$$

Solution

From the earlier example on calculating radius of curvature,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{y^3},$$

giving $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1$ and $\rho = 2\sqrt{2}$ at the point $(0.5, -1)$.



The diagram shows that the co-ordinates, (x_c, y_c), of the centre of curvature will be such that $x_c > 0.5$ and $y_c > -1$. This will be so provided that the angle, θ, is a negative acute angle; (that is, its cosine will be positive and its sine will be negative).

In fact,

$$\theta = \tan^{-1}(-1) = -45^\circ.$$

Hence,

$$\begin{aligned} x_c &= 0.5 - 2\sqrt{2}\sin(-45^\circ), \\ y_c &= -1 + 2\sqrt{2}\cos(-45^\circ). \end{aligned}$$

That is,

$$x_c = 2.5 \ \text{ and } \ y_c = 1.$$

11.4.4 EXERCISES

In the following questions, state your results in decimals correct to three places of decimals:

1. Calculate the radius of curvature at the point $(-1, 3)$ on the curve whose equation is

$$y = x + 3x^2 - x^3$$

   and hence obtain the co-ordinates of the centre of curvature.

2. Calculate the radius of curvature at the origin on the curve whose equation is

$$y = \frac{x - x^2}{1 + x^2}$$

   and hence obtain the co-ordinates of the centre of curvature.

3. Calculate the radius of curvature at the point $(1, 1)$ on the curve whose equation is

$$x^3 - 2xy + y^3 = 0$$

   and hence obtain the co-ordinates of the centre of curvature.

4. Calculate the radius of curvature at the point for which $\theta = 30°$ on the curve whose parametric equations are

$$x = 1 + \sin\theta \;\; \text{and} \;\; y = \sin\theta - \frac{1}{2}\cos 2\theta$$

   and hence obtain the co-ordinates of the centre of curvature.

11.4.5 ANSWERS TO EXERCISES

1. $\rho = 43.6705, \quad (x_c, y_c) = (42.333, 8.417)$.
2. $\rho = -1.414 \;\; (x_c, y_c) = (1, -1)$.
3. $\rho = -0.177 \;\; (x_c, y_c) = (0.875, 0.875)$.
4. $\rho = 0.590 \;\; (x_c, y_c = (-3.500, 2.750)$.