"JUST THE MATHS"

UNIT NUMBER

11.3

DIFFERENTIATION APPLICATIONS 3
(Curvature)

by

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UNIT 11.3 - DIFFERENTIATION APPLICATIONS 3

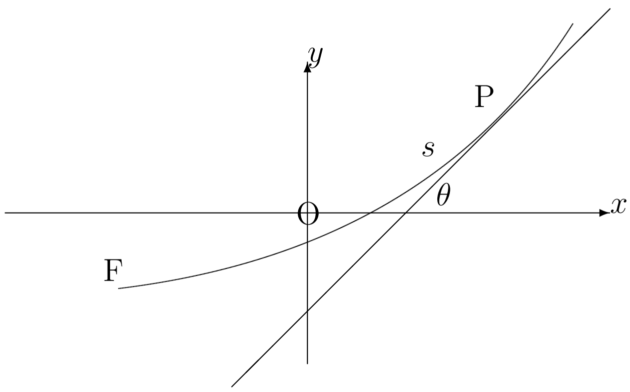
CURVATURE

11.3.1 INTRODUCTION

In the discussion which follows, consideration will be given to a method of measuring the **"tightness of bends"** on a curve. This measure will be called **"curvature"** and its definition will imply that very tight bends have large curvature.

We shall also need to distinguish between curves which are **"concave upwards"** ($\cup$), having positive curvature, and curves which are **"concave downwards"** ($\cap$), having negative curvature.

DEFINITION
Suppose we are given a curve whose equation is $y = f(x)$; and suppose that θ is the angle made with the positive x-axis by the tangent to the curve at a point, P(x, y), on it. If s is the distance to P, measured along the curve from some fixed point, F, on it then the curvature, κ, at P, is defined as the rate of increase of θ with respect to s.

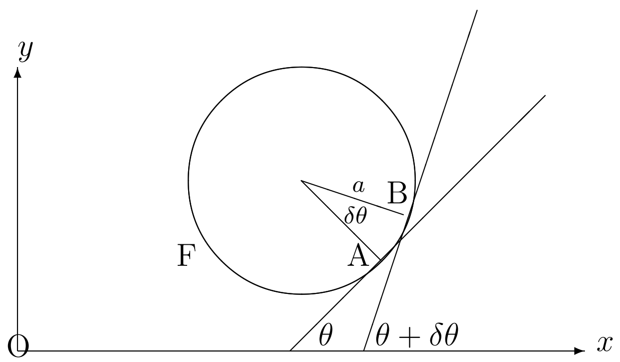


$$\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s}.$$

EXAMPLE

Determine the curvature at any point of a circle with radius a.

Solution



We shall let A be a point on the circle at which the tangent is inclined to the positive x-axis at an angle, θ, and let B be a point (close to A) at which the tangent is inclined to the positive x-axis at an angle, $\theta + \delta\theta$. The length of the arc, AB, will be called δs, where we shall assume that distances, s, are measured along the circle in a counter-clockwise sense from the fixed point, F.

The diagram shows that $\delta\theta$ is both the angle between the two tangents **and** the angle subtended at the centre of the circle by the arc, AB.

Thus, $\delta s = a\delta\theta$ which can be written

$$\frac{\delta\theta}{\delta s} = \frac{1}{a}.$$

Allowing $\delta\theta$, and hence δs, to approach zero, we conclude that

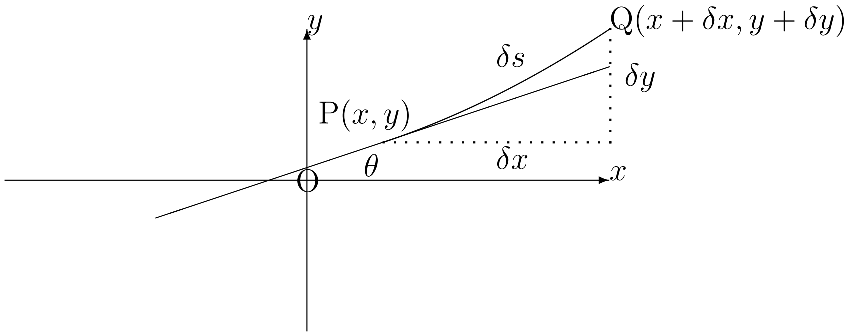
$$\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{1}{a}.$$

We note, however, that, for the lower half of the circle, θ **increases** as s increases, while, in the upper half of the circle, θ **decreases** as s increases. The curvature will therefore be positive for the lower half (which is concave upwards) and negative for the upper half (which is concave downwards).

Summary
The curvature at any point of a circle is numerically equal to the reciprocal of the radius.

11.3.2 CURVATURE IN CARTESIAN CO-ORDINATES

Given a curve whose equation is $y = f(x)$, suppose $\mathrm{P}(x, y)$ and $\mathrm{Q}(x + \delta x, y + \delta y)$ are two neighbouring points on it which are separated by a distance of δs along the curve.



In this diagram, we may observe that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \tan\theta$$

and also that

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \lim_{\delta s \to 0} \frac{\delta x}{\delta s} = \cos\theta.$$

The curvature may therefore be evaluated as follows:

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{\mathrm{d}\theta}{\mathrm{d}x}.\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\mathrm{d}\theta}{\mathrm{d}x}.\cos\theta.$$

But,

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\tan^{-1}\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \frac{1}{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}.$$

Finally,

$$\cos\theta = \frac{1}{\sec\theta} = \pm\frac{1}{\sqrt{1+\tan^2\theta}} = \pm\frac{1}{\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}};$$

and so,

$$\kappa = \pm\frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\left[1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}.$$

Notes:

(i) For a curve which is concave upwards at a particular point, the gradient, $\frac{\mathrm{d}y}{\mathrm{d}x}$, will **increase** as x increases through the point. Hence, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ will be positive at the point.

(ii) For a curve which is concave downwards at a particular point, the gradient, $\frac{\mathrm{d}y}{\mathrm{d}x}$, will **decrease** as x increases through the point. Hence, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ will be negative at the point.

(ii) In future, therefore, we may allow the value of the curvature to take the same sign as $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, giving the formula

$$\kappa = \frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\left[1+\frac{\mathrm{d}y}{\mathrm{d}x}^2\right]^{\frac{3}{2}}}.$$

EXAMPLE

Use the cartesian formula to determine the curvature at any point on the circle, centre $(0,0)$ with radius a.

Solution

The equation of the circle is

$$x^2 + y^2 = a^2,$$

which means that, for the upper half,

$$y = \sqrt{a^2 - x^2}$$

and, for the lower half,

$$y = -\sqrt{a^2 - x^2}.$$

Considering, firstly, the upper half,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{\sqrt{a^2 - x^2}}$$

and

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{\sqrt{a^2 - x^2} + \frac{x^2}{\sqrt{a^2 - x^2}}}{a^2 - x^2} = -\frac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}.$$

Therefore,

$$\kappa = \frac{-\frac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}}{\left(1 + \frac{x^2}{a^2 - x^2}\right)^{\frac{3}{2}}} = -\frac{a^2}{a^3} = -\frac{1}{a}.$$

For the lower half of the circle,

$$\kappa = \frac{1}{a}.$$

11.3.3 EXERCISES

In the following questions, state your answer in decimals correct to three places of decimals:

1. Calculate the curvature at the point $(-1,3)$ on the curve whose equation is
$$y = x + 3x^2 - x^3.$$

2. Calculate the curvature at the origin on the curve whose equation is
$$y = \frac{x - x^2}{1 + x^2}.$$

3. Calculate the curvature at the point $(1,1)$ on the curve whose equation is
$$x^3 - 2xy + y^3 = 0.$$

4. Calculate the curvature at the point for which $\theta = 30^\circ$ on the curve whose parametric equations are
$$x = 1 + \sin\theta \quad \text{and} \quad y = \sin\theta - \frac{1}{2}\cos 2\theta.$$

11.3.4 ANSWERS TO EXERCISES

1. $\kappa = 0.023$
2. $\kappa = -0.707$
3. $\kappa = -5.650$
4. $\kappa = 0.179$