"JUST THE MATHS"

UNIT NUMBER

11.2

DIFFERENTIATION APPLICATIONS 2
(Local maxima and local minima)
&
(Points of inflexion)

by

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UNIT 11.2 - APPLICATIONS OF DIFFERENTIATION 2

LOCAL MAXIMA, LOCAL MINIMA AND POINTS OF INFLEXION

11.2.1 INTRODUCTION

(a) Let us first suppose that the formula

$$s = f(t)$$

represents the distance s, travelled in time t, by a moving object from some previously chosen point on its journey.

The derivative , $\frac{\mathrm{d}s}{\mathrm{d}t}$, of s with respect to t gives the speed of the object at time t and can be represented by the slope of the tangent at the point (t, s) to the curve whose equation is $s = f(t)$.

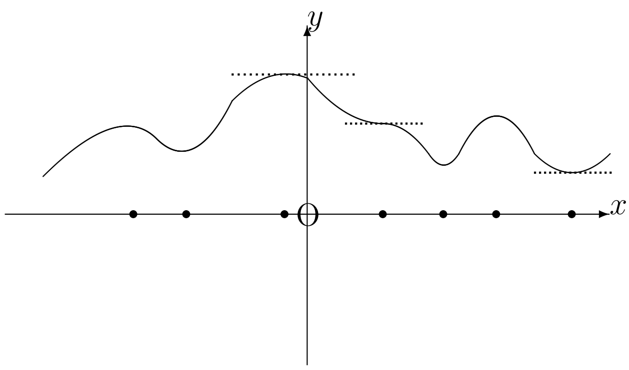
For any instant, t_0, of time, at which the object is stationary, the value of the derivative will be zero and hence, the slope of the tangent will be zero.

The corresponding point (t_0, s_0), on the graph may thus be called a **"stationary point"**.

(b) More generally, any relationship,

$$y = f(x),$$

between two variable quantities, x and y, can usually be represented by a graph of y against x and any point (x_0, y_0) on the graph at which $\frac{\mathrm{d}y}{\mathrm{d}x}$ takes the value zero is called a "stationary point". The tangent to the curve at the point (x_0, y_0) will be parallel to x-axis.



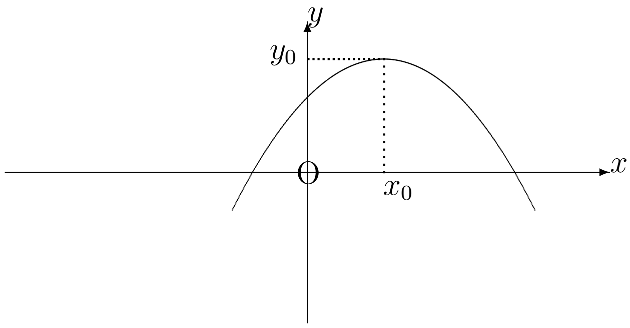
In the paragraphs which follow, we shall discuss the definitions and properties of particular kinds of stationary point.

11.2.2 LOCAL MAXIMA

A stationary point (x_0, y_0) on the graph whose equation is

$$y = f(x)$$

is said to be a **"local maximum"** if y_0 is greater than the y co-ordinates of all other points on the curve in the immediate neighbourhood of (x_0, y_0).



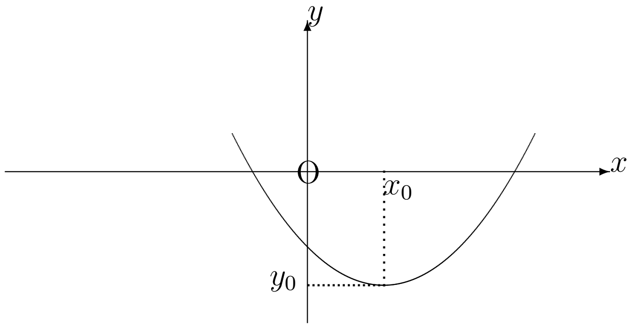
Note:
It may well happen that, for points on the curve which are some distance away from (x_0, y_0), their y co-ordinates are greater than y_0; hence, the definition of a local maximum point must refer to the behaviour of y in the immediate neighbourhood of the point.

11.2.3 LOCAL MINIMA

A stationary point (x_0, y_0) on the graph whose equation is

$$y = f(x)$$

is said to be a **"local minimum"** if y_0 is less than the y co-ordinates of all other points on the curve in the immediate neighbourhood of (x_0, y_0).



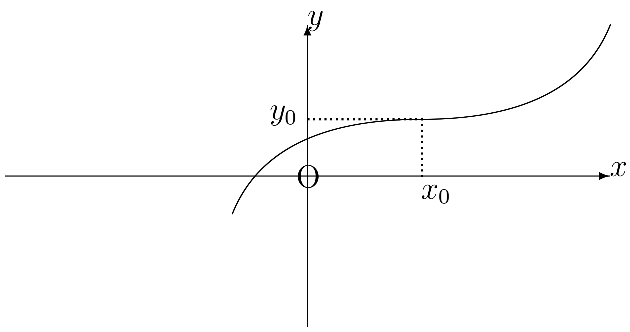
Note:
It may well happen that, for points on the curve which are some distance away from (x_0, y_0), their y co-ordinates are less than y_0; hence, the definition of a local minimum point must refer to the behaviour of y in the immediate neighbourhood of the point.

11.2.4 POINTS OF INFLEXION

A stationary point (x_0, y_0) on the graph whose equation is

$$y = f(x)$$

is said to be a **"point of inflexion"** if the curve exhibits a change in the direction bending there.



11.2.5 THE LOCATION OF STATIONARY POINTS AND THEIR NATURE

In order to determine the location of any stationary points on the curve whose equation is

$$y = f(x),$$

we simply obtain an expression for the derivate of y with respect to x, then equate it to zero. That is, we solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Having located a stationary point (x_0, y_0), we may then determine whether it is a local maximum, a local minimum, or a point of inflexion using two alternative methods. These methods will be illustrated by examples:

METHOD 1. - The "First Derivative" Method

Suppose ϵ denotes a number which is relatively small compared with x_0.

If we examine the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$, first at $x = x_0 - \epsilon$ and then at $x = x_0 + \epsilon$, the following conclusions may be drawn:

(a) If the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes from positive to negative, there is a local maximum at (x_0, y_0).

(b) If the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes from negative to positive, there is a local minimum at (x_0, y_0).

(c) If the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ does not change, there is a point of inflexion at (x_0, y_0).

EXAMPLES

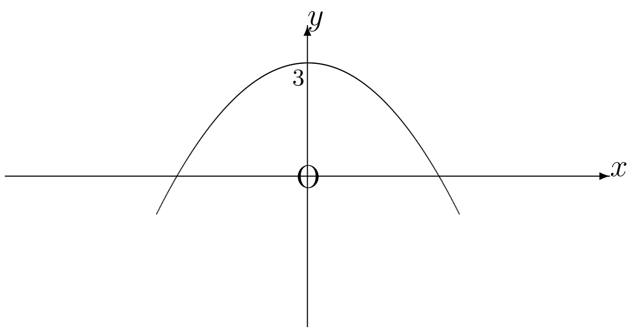
1. Determine the stationary point on the graph whose equation is

$$y = 3 - x^2.$$

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2x,$$

which is equal to zero at the point where $x = 0$ and hence, $y = 3$.
If $x = 0 - \epsilon$, (for example, $x = -0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ and
If $x = 0 + \epsilon$, (for example $x = 0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$.
Hence, there is a local maximum at the point $(0, 3)$.



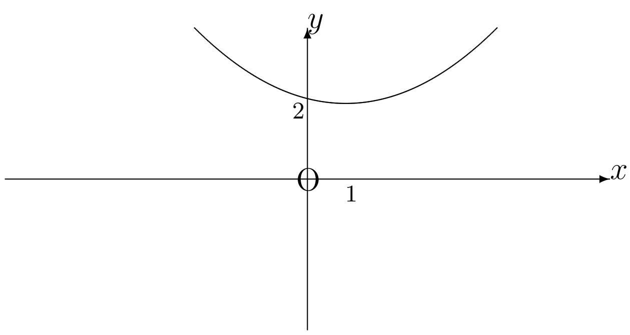
2. Determine the stationary point on the graph whose equation is

$$y = x^2 - 2x + 3.$$

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 2,$$

which is equal to zero at the point where $x = 1$ and hence, $y = 2$.
If $x = 1 - \epsilon$, (for example, $x = 1 - 0.01 = 0.99$), then $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ and
If $x = 1 + \epsilon$, (for example, $x = 1 + 0.01) = 1.01$, then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$.
Hence, there is a local minimum at the point $(1, 2)$.



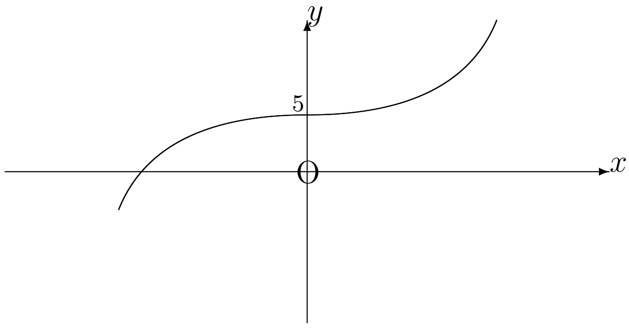
3. Determine the stationary point on the graph whose equation is

$$y = 5 + x^3.$$

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2,$$

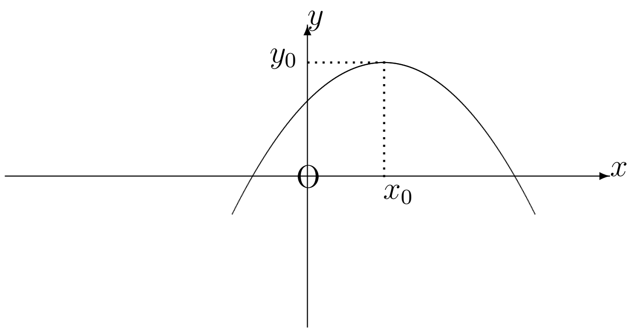
which is equal to zero at the point where $x = 0$ and hence, $y = 5$.
If $x = 0 - \epsilon$, (for example, $x = -0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ and
If $x = 0 + \epsilon$, (for example, $x = 0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$.
Hence, there is a point of inflexion at $(0, 5)$.



METHOD 2. - The "Second Derivative" Method

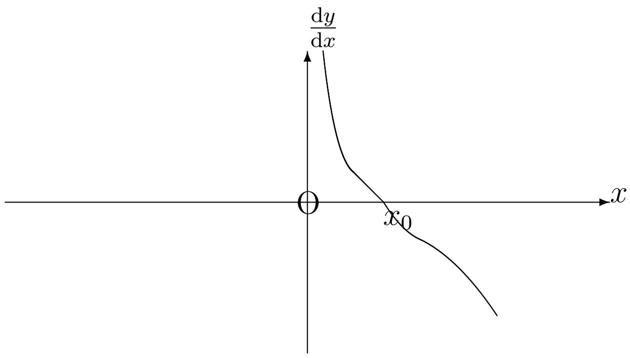
This method considers the general appearance of the graph of $\frac{\mathrm{d}y}{\mathrm{d}x}$ against x, which is called the **"first derived curve"**. The properties of the first derived curve in the neighbourhood of a stationary point (x_0, y_0) may be used to predict the nature of this point.

(a) Local Maxima



As x passes from values below x_0 to values above x_0, the corresponding values of $\frac{dy}{dx}$ steadily decrease from large positive values to large negative values, passing through zero when $x = x_0$.

This suggests that the first derived curve exhibits a **"going downwards"** tendency at $x = x_0$.



It may be expected, therefore, that the slope at $x = x_0$ on the first derived curve is **negative**. In other words,

$$\frac{d^2y}{dx^2} < 0 \ \text{ at } \ x = x_0.$$

This is the second derivative test for a local maximum.

EXAMPLE

For the curve whose equation is

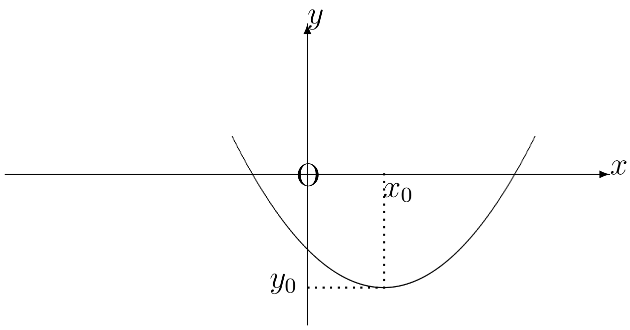
$$y = 3 - x^2,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2x \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2.$$

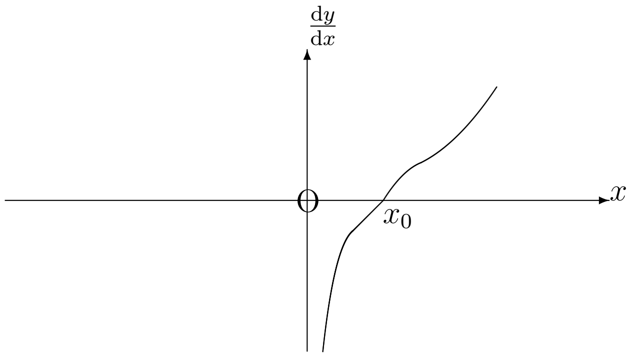
The second derivative is negative everywhere, so it is certainly negative at the stationary point $(0, 3)$ obtained in the previous method. Hence, $(0, 3)$ is a local maximum.

(b) Local Minima



As x passes from values below x_0 to values above x_0, the corresponding values of $\frac{\mathrm{d}y}{\mathrm{d}x}$ steadily increase from large negative values to large positive values, passing through zero when $x = x_0$.

This suggests that the first derived curve exhibits a **"going upwards"** tendency at $x = x_0$.



It may be expected, therefore, that the slope at $x = x_0$ on the first derived curve is **positive**. In other words,

$$\frac{d^2y}{dx^2} > 0 \quad \text{at} \quad x = x_0.$$

This is the second derivative test for a local minimum.

EXAMPLE

For the curve whose equation is

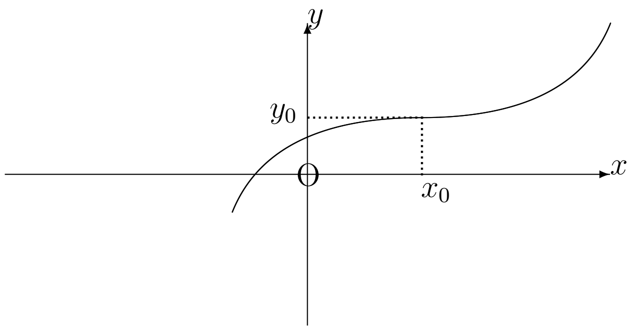
$$y = x^2 - 2x + 3,$$

we have

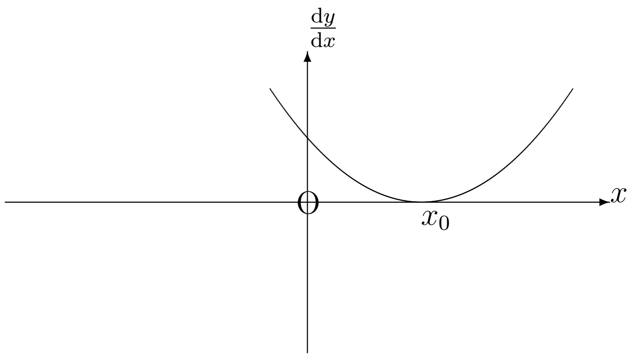
$$\frac{dy}{dx} = 2x - 2 \quad \text{and} \quad \frac{d^2y}{dx^2} = 2.$$

The second derivative is positive everywhere, so it is certainly positive at the stationary point $(1, 2)$ obtained in the previous method. Hence, $(1, 2)$ is a local minimum.

(c) Points of inflexion



As x passes from values below x_0 to values above x_0, the corresponding values of $\frac{dy}{dx}$ appear to reach either a minimum or a maximum value at $x = x_0$.



It may be expected, therefore, that the slope at $x = x_0$ on the first derived curve is zero and changes sign as x passes through the value x_0.

$$\frac{d^2y}{dx^2} = 0 \text{ at } x = x_0 \text{ and changes sign.}$$

This is the second derivative test for a point of inflexion.

EXAMPLE

For the curve whose equation is

$$y = 5 + x^3,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x.$$

The second derivative is zero when $x = 0$ and changes sign as x passes through the value zero.

Hence, the stationary point $(0, 5)$ found previously is a point of inflexion.

Notes:
(i) For a stationary point of inflexion, it is not enough that

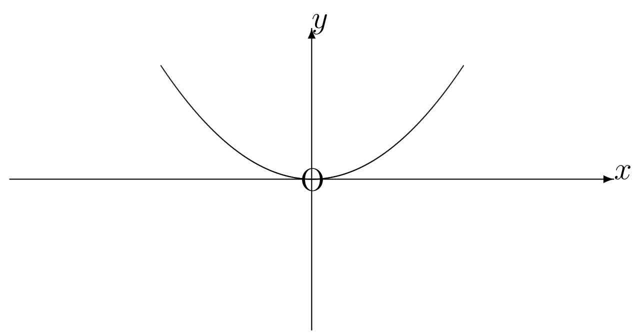
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$$

without also the change of sign.

For example, the curve whose equation is

$$y = x^4$$

is easily shown (by Method 1) to have a local minimum at the point $(0, 0)$; and yet, for this curve, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at $x = 0$.



(ii) Some curves contain what are called **"ordinary points of inflexion"**. They are not stationary points and hence, $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$; but the rest of the condition for a point of inflexion still holds. That is,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \quad \text{and} \quad \text{changes} \quad \text{sign.}$$

EXAMPLE

For the curve whose equation is

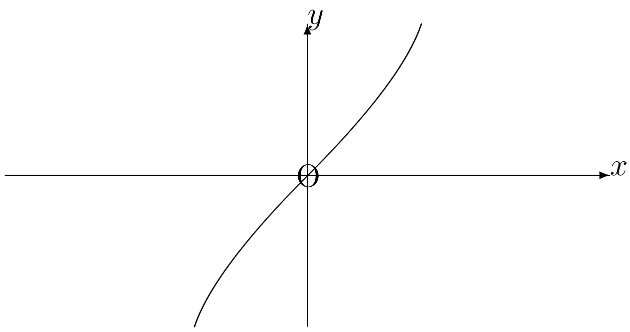
$$y = x^3 + x,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 1 \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x.$$

Hence, there are no stationary points at all; but $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at $x = 0$ and changes sign as x passes through $x = 0$.

That is, there is an ordinary point of inflexion at $(0, 0)$.



Notes:
(i) In any interval of the x-axis, the greatest value of a function of x will be either the greatest maximum or possibly the value at one end of the interval. Similarly, the least value of the function will be either the smallest minimum or possibly the value at one end of the interval.

(ii) In sketching a curve whose maxima, minima and points of inflexion are known, it may also be necessary to determine, from the equation of the curve, its points of intersection with the axes of reference.

11.2.6 EXERCISES

1. Determine the local maxima, local minima and points of inflexion (including ordinary points of inflexion) on the curves whose equations are given in the following:

   (a)
   $$y = x^3 - 6x^2 + 9x + 6;$$

   (b)
   $$y = x + \frac{1}{x}.$$

   In each case, give also a sketch of the curve.

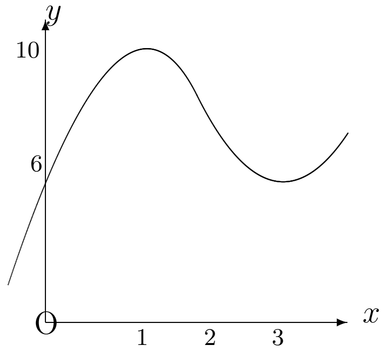
2. Show that the curve whose equation is
   $$y = \frac{1}{2x+1} + \ln(2x+1)$$
   has a local minimum at a point on the y-axis.

3. The horse-power, P, transmitted by a belt is given by
   $$P = k\left[Tv - \frac{wv^3}{g}\right],$$
   where k is a constant, v is the speed of the belt, T is the tension on the driving side and w is the weight per unit length of the belt. Determine the speed for which the horse-power is a maximum.

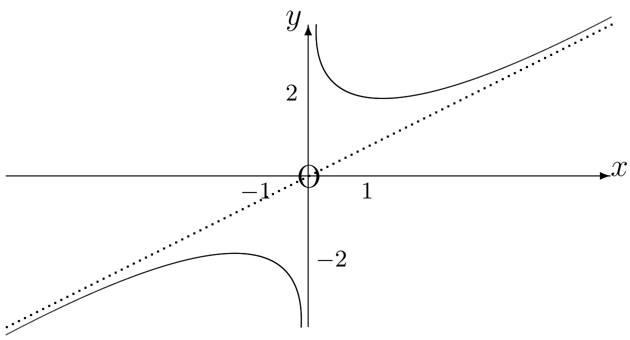
4. For x lying in the interval $-3 \leq x \leq 5$, determine the least and greatest values of the function
   $$x^3 - 12x + 20$$

11.2.7 ANSWERS TO EXERCISES

1. (a) Local maximum at $(1, 10)$, local minumum at $(3, 6)$, ordinary point of inflexion at $(2, 8)$;



   (b) Local maximum at $(-1, -2)$, local minimum at $(1, 2)$.



2. Local minimum at the point $(0, 1)$.
3. The horse-power is maximum when

$$v = \sqrt{\frac{gT}{2w}}.$$

4. The greatest value is 85 at $(5, 85)$; the least value is 4 at $(2, 4)$.