"JUST THE MATHS"

UNIT NUMBER

11.1

DIFFERENTIATION APPLICATIONS 1
(Tangents and normals)

by

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UNIT 11.1 - APPLICATIONS OF DIFFERENTIATION 1

TANGENTS AND NORMALS

11.1.1 TANGENTS

In the definition of a derivative (Unit 10.2), it is explained that the derivative of the function $f(x)$ can be interpreted as the gradient of the tangent to the curve $y = f(x)$ at the point (x, y).

We may now use this information, together with the geometry of the straight line, in order to determine the equation of the tangent to a given curve at a particular point on it.

We illustrate with examples which will then be used also in the subsequent paragraph dealing with normals.

EXAMPLES

1. Determine the equation of the tangent at the point $(-1, 2)$ to the curve whose equation is
$$y = 2x^3 + 5x^2 - 2x - 3.$$
Solution
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 + 10x - 2,$$
which takes the value -6 when $x = -1$.
Hence the tangent is the straight line passing through the point $(-1, 2)$ having gradient -6. Its equation is therefore
$$y - 2 = -6(x + 1).$$
That is,
$$6x + y + 4 = 0.$$

2. Determine the equation of the tangent at the point $(2, -2)$ to the curve to the curve whose equation is
$$x^2 + y^2 + 3xy + 4 = 0.$$
Solution
$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 3\left[x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right] = 0.$$
That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x+3y}{3x+2y},$$

which takes the value -2 at the point $(2,-2)$.

Hence, the equation of the tangent is

$$y+2 = -2(x-2).$$

That is,

$$2x+y-2=0.$$

3. Determine the equation of the tangent at the point where $t=2$ to the curve given parametrically by

$$x = \frac{3t}{1+t} \quad \text{and} \quad y = \frac{t^2}{1+t}.$$

Solution

We note first that the point at which $t=2$ has co-ordinates $\left(2, \frac{4}{3}\right)$.

Furthermore,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3}{(1+t)^2} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{2t+t^2}{(1+t)^2},$$

by the quotient rule.

Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2t+t^2}{3},$$

which takes the value $\frac{8}{3}$ when $t=2$.

Hence, the equation of the tangent is

$$y - \frac{4}{3} = \frac{8}{3}(x-2).$$

That is,

$$3y+12 = 8x.$$

11.1.2 NORMALS

The normal to a curve at a point on it is defined to be a straight line passing through this point and perpendicular to the tangent there.

Using previous work on perpendicular lines (Unit 5.2), if the gradient of the tangent is m, then the gradient of the normal will be $-\frac{1}{m}$.

EXAMPLES

In the examples of section 11.1.1, therefore, the normals to each curve at the point given will have equations as follows:

1.
$$y - 2 = \frac{1}{6}(x + 1).$$

   That is,

$$6y = x + 13.$$

2.
$$y + 2 = \frac{1}{2}(x - 2).$$

   That is,

$$2y = x - 6.$$

3.
$$y - \frac{4}{3} = -\frac{3}{8}(x - 2).$$

   That is,

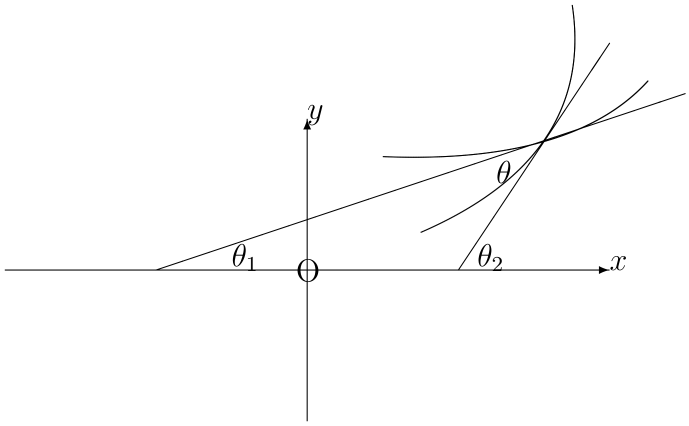
$$24y + 9x = 50.$$

Note:
It may occasionally be required to determine the angle, θ, between two curves at one of their points of intersection. This is defined to be the angle between the tangents at this point; and, if the gradients of the tangents are $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$, then the angle $\theta \equiv \theta_2 - \theta_1$ and is given by

$$\tan\theta = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2\tan\theta_1}.$$

That is,

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}.$$



11.1.3 EXERCISES

1. Determine the equations of the tangent and normal to the following curves at the point given:

   (a)

   $$8y = x^3 \quad \text{at} \quad (2, 1);$$

   (b)

   $$y = \frac{e^{2x}\cos x}{(1+x)^3} \quad \text{at} \quad (0, 1).$$

2. The parametric equations of a curve are

   $$x = 1 + \sin 2t, \quad y = 1 + \cos t + \cos 2t.$$

   Determine the equation of the tangent to the curve at the point for which $t = \frac{\pi}{2}$.

3. Determine the quation of the tangent at the point $(2, 3)$ to the curve whose equation is

$$3x^2 + 2y^2 = 30.$$

4. Determine the equation of the normal at the point $(-1, 2)$ to the curve whose equation is

$$2xy + 3xy^2 - x^2 + y^3 + 9 = 0.$$

11.1.4 ANSWERS TO EXERCISES

1. (a) The tangent is

$$2y = 3x - 4,$$

   and the normal is

$$2x + 3y = 7;$$

   (b) The tangent is

$$y = 1 - x,$$

   and the normal is

$$y = x + 1.$$

2. The tangent is

$$2y = x - 1.$$

3. The tangent is

$$x + y = 5.$$

4. The normal is

$$x + 9y = 17.$$