"JUST THE MATHS"

UNIT NUMBER

10.4

DIFFERENTIATION 4
(Products and quotients)
&
(Logarithmic differentiation)

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UNIT 10.4 - DIFFERENTIATION 4

PRODUCTS, QUOTIENTS AND LOGARITHMIC DIFFERENTIATION

10.4.1 PRODUCTS

Suppose

$$y = u(x)v(x),$$

where $u(x)$ and $v(x)$ are two functions of x.

Suppose, also, that a small increase of δx in x gives rise to increases (positive or negative) of δu in u, δv in v and δy in y.

Then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{(u + \delta u)(v + \delta v) - uv}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{uv + u\delta v + v\delta u + \delta u\delta v - uv}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} \right].$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[u.v] = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Hint: Think of this as
(FIRST x DERIVATIVE OF SECOND) + (SECOND x DERIVATIVE OF FIRST)

EXAMPLES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = x^7 \cos 3x.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^7. - 3\sin 3x + \cos 3x.7x^6 = x^6[7\cos 3x - 3x\sin 3x].$$

2. Evaluate $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = -1$ in the case when

$$y = (x^2 - 8)\ln(2x + 3).$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x^2 - 8).\frac{1}{2x + 3}.2 + \ln(2x + 3).2x = 2\left[\frac{x^2 - 8}{2x + 3} + x\ln(2x + 3)\right].$$

When $x = -1$, this has value -14 since $\ln 1 = 0$.

10.4.2 QUOTIENTS

Suppose, this time, that

$$y = \frac{u(x)}{v(x)}.$$

Then, we may write

$$y = u(x).[v(x)]^{-1}$$

in order to use the rule already known for products.

We obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u.(-1)[v]^{-2}.\frac{\mathrm{d}v}{\mathrm{d}x} + v^{-1}.\frac{\mathrm{d}u}{\mathrm{d}x},$$

which can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{u}{v}\right] = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}.$$

EXAMPLES

1. Using the formula for the derivative of a quotient, show that the derivative with respect to x of the function $\tan x$ is the function $\sec^2 x$.

   Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan x] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin x}{\cos x}\right] = \frac{(\cos x).(\cos x) - (\sin x).(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = \frac{2x+1}{(5x-3)^3}.$$

Solution

Using $u(x) \equiv 2x + 1$ and $v(x) \equiv (5x-3)^3$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(5x-3)^3.2 - (2x+1).3(5x-3)^2.5}{(5x-3)^6}.$$

The expression $(5x-3)^2$ may be cancelled as a common factor of both numerator and denominator, leaving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(5x-3).2 - 15(2x+1)}{(5x-3)^4} = -\frac{20x+21}{(5x-3)^4}.$$

Note:

The step in the second example above, where a common factor could be cancelled, may be avoided if we use a modified version of the rule for quotients when the function can be considered in the form

$$\frac{u}{v^n}.$$

It can be shown that, if

$$y = \frac{u}{v^n},$$

then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - nu\frac{\mathrm{d}v}{\mathrm{d}x}}{v^{n+1}}.$$

For instance, in Example 2 above, we could write

$$u \equiv 2x+1 \quad v \equiv 5x-3 \quad \text{and} \quad n = 3$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(5x-3).2 - 3(2x+1).5}{(5x-3)^4},$$

as before.

10.4.3 LOGARITHMIC DIFFERENTIATION

The algebraic properties of natural logarithms (see Unit 1.4), together with the standard derivative of $\ln x$ and the rules of differentiation, enable us to differentiate two specific kinds of function as described below:

(a) Functions containing a variable index

The most familiar function with which to introduce this technique is the **"exponential function"**, e^x.

Suppose we let

$$y = e^x;$$

then, by properties of natural logarithms, we can write

$$\ln y = x;$$

and, if we differentiate both sides **with respect to** x, we obtain

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 1.$$

That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y = e^x.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^x\right] = e^x.$$

Notes:

(i) After taking logarithms, we could have differentiated the statement $x = \ln y$ with respect to y, obtaining

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{y}.$$

But it can be shown that, for most functions,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$$

so that the same result is obtained as before.

(ii) The derivative of e^x may easily be used to establish the standard derivatives of the hyperbolic functions, $\sinh x$, $\cosh x$ and $\tanh x$ as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sinh x] = \cosh x, \quad \frac{\mathrm{d}}{\mathrm{d}x}[\cosh x] = \sinh x, \quad \frac{\mathrm{d}}{\mathrm{d}x}[\tanh x] = \mathrm{sech}^2\ x.$$

The first two of these follow from the definitions

$$\sinh x \equiv \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x \equiv \frac{e^x + e^{-x}}{2},$$

while the third may be obtained using the definition

$$\tanh x \equiv \frac{\sinh x}{\cosh x},$$

together with the Quotient Rule.

FURTHER EXAMPLES

1. Write down the derivative with respect to x of the function

$$e^{\sin x}$$

Solution

All that is required in this example is the standard derivative of e^x together with the Function of a Function Rule. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{\sin x}\right] = e^{\sin x}.\cos x.$$

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = (3x + 2)^x.$$

Solution

Taking natural logarithms of both sides,

$$\ln y = x \ln(3x + 2).$$

Differentiating both sides with respect to x and using the Product Rule gives

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x.\frac{3}{3x+2} + \ln(3x+2).1.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (3x+2)^x\left[\frac{3x}{3x+2} + \ln(3x+2)\right].$$

(b) Products or Quotients with more than two elements

We have already discussed the rules for differentiating products and quotients; but, in certain cases, it is easier to make use of logarithmic differentiation. Essentially, we use this alternative method when a product or a quotient involves more than the two functions $u(x)$ and $v(x)$ mentioned earlier.

We illustrate with examples:

EXAMPLES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = e^{x^2}.\cos x.(x+1)^5.$$

Solution

Taking natural logarithms of both sides,

$$\ln y = x^2 + \ln(\cos x) + 5\ln(x+1).$$

Differentiating both sides with respect to x,

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - \frac{\sin x}{\cos x} + \frac{5}{x+1}.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x^2}.\cos x.(x+1)^5\left[2x - \tan x + \frac{5}{x+1}\right].$$

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = \frac{e^x.\sin x}{(7x+1)^4}.$$

Solution

Taking natural logarithms of both sides,

$$\ln y = x + \ln(\sin x) - 4\ln(7x+1).$$

Differentiating both sides with respect to x,

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{\cos x}{\sin x} - 4.\frac{7}{7x+1}.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x.\sin x}{(7x+1)^4}\left[1 + \cot x - \frac{28}{7x+1}\right].$$

Note:
In all examples on logarithmic differentiation, the original function will appear as a factor at the beginning of its derivative.

10.4.4 EXERCISES

1. Differentiate the following functions with respect to x:

   (a)
   $$\sin x.\cos x;$$

   (b)
   $$(x^2+3).\sin 2x;$$

   (c)
   $$x.(x^2+1)^{\frac{1}{2}};$$

   (d)
   $$x^2\ln(1-2x).$$

2. Differentiate the following functions with respect to x:

   (a)
   $$\frac{\cos x}{\sin x} \quad (\text{that is,} \quad \cot x);$$

(b)

$$\frac{x^2-2}{(x+1)^2};$$

(c)

$$\frac{\cos x+\sin x}{\cos x-\sin x};$$

(d)

$$\frac{x}{(2x-x^2)^{\frac{1}{2}}}.$$

3. Differentiate the following functions with respect to x:

(a)

$$e^{x^2+1};$$

(b)

$$e^{1-x-x^2};$$

(c)

$$(2x+1)e^{4-x^3};$$

(d)

$$\frac{e^{1-7x}}{3x+2};$$

(e)

$$x.\sinh(x^2+1);$$

(f)

$$\operatorname{sech} x.$$

4. Use logarithms to differentiate the following functions with respect to x:

(a)

$$a^x \quad (a \text{ constant});$$

(b)

$$(x^2+1)^{3x};$$

(c)

$$(\sin x)^x;$$

(d)

$$\frac{x(x-2)}{(x+1)(x+3)};$$

(e)

$$\frac{e^{2x}.\ln x}{(x-1)^3}.$$

10.4.5 ANSWERS TO EXERCISES

1. (a)

$$\cos^2 x - \sin^2 x \quad (\text{or} \quad \cos 2x);$$

(b)

$$2(x^2+3)\cos 2x + 2x \sin 2x;$$

(c)

$$\frac{2x^2+1}{(x^2+1)^{\frac{1}{2}}};$$

(d)

$$2x\ln(1-2x) - \frac{2x^2}{1-2x}.$$

2. (a)

$$-\text{cosec}^2 x;$$

(b)

$$\frac{4+2x}{(x+1)^3};$$

(c)

$$\frac{2}{(\cos x - \sin x)^2};$$

(d)

$$\frac{x}{(2x - x^2)^{\frac{3}{2}}}.$$

3. (a)

$$2xe^{x^2+1};$$

(b)

$$-(1+2x)e^{1-x-x^2};$$

(c)

$$2.e^{4-x^3} - 3x^2(2x+1)e^{4-x^3};$$

(d)

$$-\frac{e^{1-7x}.(21x+17)}{(3x+2)^2};$$

(e)

$$\sinh(x^2+1) + 2x^2\cosh(x^2+1);$$

(f)

$$-\operatorname{cosech}^2 x.$$

4. (a)

$$a^x.\ln a;$$

(b)

$$(x^2+1)^{3x}\left[3\ln(x^2+1) + \frac{6x^2}{x^2+1}\right];$$

(c)

$$(\sin x)^x\left[\ln\sin x + x\cot x\right];$$

(d)

$$\frac{x(x-2)}{(x+1)(x+3)}\left[\frac{1}{x} + \frac{1}{x-2} - \frac{1}{x+1} - \frac{1}{x+3}\right];$$

(e)

$$\frac{e^{2x}.\ln x}{(x-1)^3}\left[2 + \frac{1}{x\ln x} - \frac{3}{x-1}\right].$$