"JUST THE MATHS"

UNIT NUMBER

10.3

DIFFERENTIATION 3
(Elementary techniques of differentiation)

by

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UNIT 10.3 - DIFFERENTIATION 3

ELEMENTARY TECHNIQUES OF DIFFERENTIATION

10.3.1 STANDARD DERIVATIVES

In Unit 10.2, reference was made to the use of a table of standard derivatives and such a table can be found in the appendix at the end of this Unit.

However, for the time being, a very short list of standard derivatives is all that is necessary since other derivatives may be developed from them using techniques to be discussed later in this and subsequent Units.

$f(x)$	$f'(x)$
a const.	0
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\ln x$	$\frac{1}{x}$

Note:
In the work which now follows, standard derivatives may be used which have not, here, been obtained from first principles; but the student is expected to be able to quote results from a table of derivatives including those for which no proof has been given.

10.3.2 RULES OF DIFFERENTIATION

(a) Linearity

Suppose $f(x)$ and $g(x)$ are two functions of x while A and B are constants. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[Af(x)+Bg(x)\right]=A\frac{\mathrm{d}}{\mathrm{d}x}[f(x)]+B\frac{\mathrm{d}}{\mathrm{d}x}[g(x)].$$

Proof:

The left-hand side is equivalent to

$$\lim_{\delta x\to 0}\frac{[Af(x+\delta x)+Bg(x+\delta x)]-[Af(x)+Bg(x)]}{\delta x}$$

$$=A\left[\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}\right]+B\left[\lim_{\delta x\to 0}\frac{g(x+\delta x)-g(x)}{\delta x}\right]$$

$$= A\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] + B\frac{\mathrm{d}}{\mathrm{d}x}[g(x)].$$

The result, so far, deals with a **"linear combination"** of **two** functions of x but is easily extended to linear combinations of **three or more** functions of x.

EXAMPLES

1. Write down the expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = 6x^2 + 2x^6 + 13x - 7.$$

Solution

Using the linearity property, the standard derivative of x^n, and the derivative of a constant, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6\frac{\mathrm{d}}{\mathrm{d}x}[x^2] + 2\frac{\mathrm{d}}{\mathrm{d}x}[x^6] + 13\frac{\mathrm{d}}{\mathrm{d}x}[x^1] - \frac{\mathrm{d}}{\mathrm{d}x}[7]$$

$$= 12x + 12x^5 + 13.$$

2. Write down the derivative with respect to x of the function

$$\frac{5}{x^2} - 4\sin x + 2\ln x.$$

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{5}{x^2} - 4\sin x + 2\ln x\right]$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left[5x^{-2} - 4\sin x + 2\ln x\right]$$

$$= -10x^{-3} - 4\cos x + \frac{2}{x}$$

$$= \frac{-10}{x^3} - 4\cos x + \frac{2}{x}.$$

(b) Composite Functions (or Functions of a Function)

(i) Functions of a Linear Function

Expressions such as $(5x+2)^{16}$, $\sin(2x+3)$ and $\ln(7-4x)$ may be called **"functions of a linear function"** and have the general form

$$f(ax+b),$$

where a and b are constants. The function $f(x)$ would, of course, be the one obtained on replacing $ax+b$ by a single x; hence, in the above illustrations, $f(x)$ would be x^{16}, $\sin x$ and $\ln x$, respectively.

Functions of a linear function may be differentiated as easily as $f(x)$ itself on the strength of the following argument:

Suppose we write

$$y = f(u) \quad \text{where} \quad u = ax+b.$$

Suppose, also, that a small increase of δx in x gives rise to increases (positive or negative) of δy in y and δu in u. Then:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\delta y}{\delta u}\frac{\delta u}{\delta x}.$$

Assuming that δy and δu tend to zero as δx tends to zero, we can say that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta u\to 0}\frac{\delta y}{\delta u} \times \lim_{\delta x\to 0}\frac{\delta u}{\delta x}.$$

That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x}.$$

This rule is called the **"Function of a Function Rule"** or **"Composite Function Rule"** or **"Chain Rule"** and has applications to a much wider class of composite functions than has so far been discussed. But, for the moment we restrict the discussion to functions of a linear function.

EXAMPLES

1. Determine $\frac{dy}{dx}$ when $y = (5x + 2)^{16}$.

   Solution

   First, we write $y = u^{16}$ where $u = 5x + 2$.

   Then, $\frac{dy}{du} = 16u^{15}$ and $\frac{du}{dx} = 5$.

   Hence, $\frac{dy}{dx} = 16u^{15}.5 = 80(5x + 2)^{15}$.

2. Determine $\frac{dy}{dx}$ when $y = \sin(2x + 3)$.

   Solution

   First, we write $y = \sin u$ where $u = 2x + 3$.

   Then, $\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 2$.

   Hence, $\frac{dy}{dx} = \cos u.2 = 2\cos(2x + 3)$.

3. Determine $\frac{dy}{dx}$ when $y = \ln(7 - 4x)$.

   Solution

   First, we write $y = \ln u$ where $u = 7 - 4x$.

   Then, $\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = -4$.

   Hence, $\frac{dy}{dx} = \frac{1}{u}.(-4) = \frac{-4}{7-4x}$.

Note:
It is hoped that the student will quickly appreciate how the fastest way to obtain the derivative of a function of a linear function is to treat the expression $ax + b$ initially as if it were a single x; then, multiply the final result by the constant value, a.

(ii) Functions of a Function in general

The formula

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

is in no way dependent on the fact that the examples so far used to illustrate it have involved functions of a linear function. Exactly the same formula may be used for the composite function

$$f[g(x)],$$

whatever the functions $f(x)$ and $g(x)$ happen to be. All we need to do is to write

$$y = f(u) \quad \text{where} \quad u = g(x),$$

then apply the formula.

EXAMPLES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

   $$y = (x^2 + 7x - 3)^4.$$

   Solution

   Letting $y = u^4$ where $u = x^2 + 7x - 3$, we have

   $$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x} = 4u^3.(2x + 7) \\ &= 4(x^2 + 7x - 3)^3(2x + 7). \end{aligned}$$

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

   $$y = \ln(x^2 - 3x + 1).$$

   Solution

   Letting $y = \ln u$ where $u = x^2 - 3x + 1$, we have

   $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{u}.(2x - 3) = \frac{2x - 3}{x^2 - 3x + 1}.$$

3. Determine the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 1$ in the case when

   $$y = 2\sin(5x^2 - 1) + 19x.$$

   Solution

   Consider, first, the function $2\sin(5x^2 - 1)$ which we shall call z.
   Its derivative is $\frac{\mathrm{d}z}{\mathrm{d}x}$, where $z = 2\sin(5x^2 - 1)$.

Let $z = 2\sin u$ where $u = 5x^2 - 1$; then,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x} = 2\cos u.10x = 20x\cos(5x^2 - 1).$$

Hence, the complete derivative is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 20x\cos(5x^2 - 1) + 19.$$

Finally, when $x = 1$, this derivative has the value $20\cos 4 + 19$, which is approximately equal to 5.927, remembering that the calculator must be in **radian mode**.

Note:
Again, it is hoped that the student will appreciate how there is a faster way of differentiating composite functions in general. We simply treat $g(x)$ initially as if it were a single x, then multiply by $g'(x)$ afterwards.

For example,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin^3 x\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[(\sin x)^3\right] = 3(\sin x)^2.\cos x = 3\sin^2 x.\cos x.$$

10.3.3 EXERCISES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the following cases:

   (a)
   $$y = 3x^3 - 8x^2 + 11x + 9;$$

   (b)
   $$y = 10\cos x + 5\sin x - 14x^7;$$

   (c)
   $$y = (2x - 7)^5;$$

   (d)
   $$y = (2 - 5x)^{-\frac{5}{2}};$$

(e)

$$y = \sin\left(\frac{\pi}{2} - x\right); \text{ that is, } \cos x;$$

(f)

$$y = \cos(4x + 1);$$

(g)

$$y = \ln(4 - 2x);$$

(h)

$$y = \ln\left[\frac{3x - 8}{6x + 2}\right].$$

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the cases when

(a)

$$y = (4 - 7x^3)^8;$$

(b)

$$y = (x^2 + 1)^{\frac{3}{2}};$$

(c)

$$y = \cos^5 x;$$

(d)

$$y = \ln(\ln x).$$

3. If $y = \sin(\cos x)$, evaluate $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = \frac{\pi}{2}$.

4. If $y = \cos(7x^5 - 3)$, evaluate $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 1$.

10.3.4 ANSWERS TO EXERCISES

1. (a)

$$9x^2 - 16x + 11;$$

(b)

$$-10\sin x + 5\cos x - 98x^6;$$

(c)

$$10(2x-7)^4;$$

(d)

$$\frac{25}{2}(2-5x)^{-\frac{7}{2}};$$

(e)

$$-\cos\left(\frac{\pi}{2}-x\right);\ \text{that is,}\ \ -\sin x;$$

(f)

$$-4\sin(4x+1);$$

(g)

$$\frac{-2}{4-2x}\ \text{or}\ \frac{2}{2x-4};$$

(h)

$$\frac{3}{3x-8}-\frac{6}{6x+2}=\frac{54}{(3x-8)(6x+2)}.$$

2. (a)

$$-168x^2(4-7x^3)^7;$$

(b)

$$3x(x^2+1)^{\frac{1}{2}};$$

(c)

$$-5\cos^4 x.\sin x;$$

(d)

$$\frac{1}{x\ln x}.$$

3.

$$-1$$

4.

$$-35\sin 4\cong 26.488$$

APPENDIX - A Table of Standard Derivatives

$f(x)$	$f'(x)$
a (const.)	0
x^n	nx^{n-1}
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$
$\tan ax$	$a\text{sec}^2 ax$
$\cot ax$	$-a\text{cosec}^2\ ax$
$\text{sec}\ ax$	$a\text{sec}\ ax.\tan ax$
$\text{cosec}\ ax$	$-a\text{cosec}\ ax.\cot ax$
$\ln x$	$1/x$
e^{ax}	ae^{ax}
a^x	$a^x.\ln a$
$\sinh ax$	$a\cosh ax$
$\cosh ax$	$a\sinh ax$
$\tanh ax$	$a\text{sech}^2\ ax$
$\text{sech}\ ax$	$-a\text{sech}\ ax.\tanh ax$
$\text{cosech}\ ax$	$-a\text{cosech}\ ax.\coth x$
$\ln(\sin x)$	$\cot x$
$\ln(\cos x)$	$-\tan x$
$\ln(\sinh x)$	$\coth x$
$\ln(\cosh x)$	$\tanh x$
$\sin^{-1}(x/a)$	$1/\sqrt{(a^2-x^2)}$
$\cos^{-1}(x/a)$	$-1/\sqrt{(a^2-x^2)}$
$\tan^{-1}(x/a)$	$a/(a^2+x^2)$
$\sinh^{-1}(x/a)$	$1/\sqrt{(x^2+a^2)}$
$\cosh^{-1}(x/a)$	$1/\sqrt{(x^2-a^2)}$
$\tanh^{-1}(x/a)$	$a/(a^2-x^2)$